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Research Article

Exploring Nonlinear Dynamics and Solitary Waves in the Fractional Klein–Gordon Model

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Various nonlinear evolution equations reveal the inner characteristics of numerous real-life complex phenomena. Using the extended fractional Riccati expansion method, we investigate optical soliton solutions of the fractional Klein–Gordon equation within this modified framework. This model in physics claims the existence of energy particles and defines the relativistic wave. The proposed procedures provide insight into wave spread and optical solitons, which enterprises in current broadcast communications can utilize to empower fast and long-distance information transmission with minimal signal degradation. They are important for their reliability and optical communication networking. This method can analytically formulate optical soliton solutions using rational, hyperbolic, and trigonometric functions. The interaction between the breather and the king wave, as well as the bright and dark bell-shaped singular soliton waves, are the numerical forms of the obtained solutions, examined using three- and two-dimensional diagrams. These solutions are obtained using the proposed method. For $[\alpha, \beta = 0.1, 0.5, 0.9]$, we illustrate the impact of conformable and beta fractional parameters in two-dimensional graphs. Understanding and clarifying the physical characteristics of waves may be made easier by the collected results. Nonlinear optics, optical communications, and engineering all rely on unique and precise soliton solutions, and the aforementioned applied techniques may, therefore, serve as a valuable tool for this purpose.

Keywords: beta fractional derivative; chaos; comfortable fractional derivative; optical solutions; soliton; the extended fractional Riccati expansion method

1. Introduction

According to the study, optical solitons may be a viable alternative to ultralong-distance and high-capacity transmission in final degradation [1]. Optical solitons have been the subject of much examination since their exploratory disclosure by Mollenauer in 1980 and their forecast by Hasegawa in 1973 [2, 3]. Waves that travel through the propagation dimension without encountering any interference are referred to as solitons, which are a type of alternative solution. During the propagation of optical waves, certain forms of interaction emerge when the distance between neighboring optical solitons decreases. In fast correspondence frameworks, this outcome is superior to transmission speed and correspondence limit. Some researchers have developed integrated communication systems that can transmit data over multiple channels using optical soliton waves [4–8]. Optical soliton arrangements are significant in media communications for their capacity to send data without twisting over significant distances. While nonlinearity introduces distortions, dispersion causes signals to spread out and degrade over a greater distance. Solitons, on the other hand, get around these obstacles by maintaining their shape and amplitude as they travel. Furthermore, solitons are extremely useful in multiplexing applications because they permit the unhindered simultaneous transmission of multiple data channels at various frequencies. In addition, undersea cables and other long-distance communication links benefit significantly from their stability and durability. In essence, optical soliton solutions form the foundation of modern telecommunications, enabling the transmission of data at high speeds over long distances with minimal signal degradation. They are significant because they have changed how optical communication networks work and are reliable. The research into optical soliton solutions has led to the development of novel approaches to problem-solving and the solid adoption of particular approaches by certain fields, such as the first integral process [9], variational iteration scheme [10], modified hyperbolic function expansion method [11], Hirota bilinear manner [12, 13], manner of Fan subequation (PFS-E) [14], subequation scheme [15], EMSE scheme [16, 17], the (G'/G)-expansion manner [18], extended method of mapping [19], a direct algebraic manner [20], a modified extended auxiliary equation mapping manner [21], a modified simple equation manner [22], the modified extended tanh manner [23], the generalized exponential procedure [24], and so on [25, 26]. Many researchers have employed various approaches to study the behavior and dynamics of solitons. The investigation of soliton dynamics and the conduct of stability analysis have added increasing importance in recent studies. For example, Rehman and Ahmad [27] constructed optical solitons in various formats and investigated the modulation instability of optical soliton pulses using linear stability theory. Multisoliton solutions of the time-fractional coupled nonlinear Schrödinger equation in optical fibers were studied by Ahmad et al. [28], revealing solitary wave profiles in terms of actual components. Akram et al. [29] conducted stability analysis and explored dispersive optical solitons of the fractional Schrödinger–Hirota equation. Youssri [30] developed spectral solutions for the nonlinear fractional Klein–Gordon equation and introduced a new operational matrix of fractional derivatives of Fibonacci polynomials in his study. A semianalytic spectral collocation procedure, as presented in the study by Youssri and Atta [31], is used to solve a fractional boundary value problem that simulates the dynamics of the human corneal shape.

Nonlinear optics, Bose–Einstein condensates, and condensed matter physics are just a few of the fields in which the $(2+1)$-dimensional NLSE finds use, and wave packet dynamics and chirality play a significant role. The following is our investigation of the fractional Klein–Gordon optical soliton solutions in this work [32]:

$$u_{tt} - u_{xx} - \psi u - \varepsilon u^3 = 0, \tag{1}$$

where ψ and ε are arbitrary constants. There are many techniques applied to the exact solution from the KG equation, such as the generalized Kudryashov method procedure and modified $(\frac{G}{G})$-expansion manner [33], modified F-expansion approaches, the ESE scheme [34], and so on. For the proposed model, we first time investigated the extended fractional Riccati expansion scheme to derive some dynamic optical wave patterns. The order of the articles is as follows: The fractional derivative is discussed in greater detail in the Section 2. The guidelines for the extended fractional Riccati expansion scheme are discussed in Section 3. The application of analytical methods in Section 4. The description of graphs is discussed, and the effect of fractional parameters is observed in Section 5. Comparisons are discussed in Section 6. The conclusion should be summed up at the end of Section 7.

2. About Fractional Derivative

The exact traveling wave solutions of nonlinear partial differential equations are raised by applying analytical as well as numerical methods. The fractal-order NLEEs are more convenient in contrast to integer-order NLEEs to depict physical phenomena [35]. Various fractional derivatives have been presented and described in fractional calculus. Some of them are narrated below.

2.1. Conformable Fractional Derivative. The conformable fractional derivative is introduced of order α with respect to t, the independent variable as follows [36]:

$$D_t^{\alpha}(x(t)) = \lim_{\varsigma\to 0} \frac{x(t+\varsigma t^{1-\alpha}) - x(t)}{\varsigma}, t>0, \alpha \in (0,1], \tag{2}$$

for the function $x = x(t) : [0, \infty) \to \mathscr{R}$. Critical fundamental properties like Tailor series expansion, the chain rule, and Laplace transformation are satisfied by a conformable fractional differential operator. This kind of properties have been brought out through the following theorems:

Theorem 1. *Consider, $m = m(t)$ and $n = n(t)$ are α-differentiable for all positive t with the order of the derivative $\alpha \in (0,1]$. Hence,*

1. $D_t^{\alpha}(s_1 m + s_2 n) = s_1 D_t^{\alpha}(m) + s_2 D_t^{\alpha}(n)$, for $\forall s_1, s_2 \in R$.
2. $D_t^{\alpha}(t^n) = nt^{n-\alpha}, \forall\, n \in R$.
3. $D_t^{\alpha}(\varphi) = 0$, for all constant function $m(t) = \varphi$.
4. $D_t^{\alpha}(mn) = mD_t^{\alpha}(n) + nD_t^{\alpha}(m)$.
5. $D_t^{\alpha}(\frac{m}{n}) = \frac{nD_t^{\alpha}(m) - mD_t^{\alpha}(n)}{n^2}$.
6. $D_t^{\alpha} m(t) = t^{1-\alpha}\frac{dm}{dt}$.

Theorem 2. *Assume that γ is differentiable and defined in the range of δ. Let consider, an δ-conformable differentiable function $q = q(t)$. Then,*

$$D_t^{\alpha}(\delta\, o\, \gamma)(t) = t^{1-\delta}\gamma'(t)\delta'(\gamma(t)).$$

2.2. Beta Fractional Derivative. Consider a function f for $f : [d, \infty) \to \mathbb{R}$. Then, the beta derivative is stated as:

$${}_0^M\mathscr{D}_t^{\beta} f(t) = \lim_{\eta\to 0} \frac{f\left(t+\eta\left(t+\frac{1}{\Gamma\beta}\right)^{1-\beta}\right) - f(t)}{\eta}, t \geq d, \beta \in (0,1].$$

Then, f is said to be beta differentiable if the above limit exists [37].

Theorem 3. *If ϑ and δ is β-differentiable, $\beta \in (0, 1]$ at t.Then,*

1. $\mathscr{D}_t^\beta(m\,\vartheta(t) + n\,\delta(t)) = m\mathscr{D}_t^\beta \vartheta(t) + n\mathscr{D}_t^\beta \delta(t)$.
2. $\mathscr{D}_t^\beta(c) = 0, c \in \mathbb{R}$.
3. $\mathscr{D}_t^\beta(\vartheta(t)\delta(t)) = \vartheta(t)\mathscr{D}_t^\beta(\delta(t)) + \delta(t)\mathscr{D}_t^\beta(\vartheta(t))$.
4. $\mathscr{D}_t^\beta(\frac{\vartheta(t)}{\delta(t)}) = \frac{\delta(t)\mathscr{D}_t^\beta(\vartheta(t)) - \vartheta(t)\mathscr{D}_t^\beta(\delta(t))}{\delta^2}$.
5. $\mathscr{D}_t^\beta(\vartheta(t)) = (t + \frac{1}{\Gamma\beta})^{1-\beta}\frac{d\vartheta(t)}{dt}$.
6. $\mathscr{D}_t^\beta(\vartheta(t) \circ \delta(t)) = (t + \frac{1}{\Gamma\beta})^{1-\beta}\delta'(t)\vartheta'(\delta(t))$.

3. Description of the Method

We would like to emphasize that our Riccati-type expansions offer several advantages over existing methods for solving nonlinear equations. First, they are more computationally efficient, allowing for faster solutions. Second, it yields higher accuracy, particularly in problems involving strong nonlinearities. Finally, these approaches are more flexible in modeling various physical phenomena and are particularly useful in applications within areas such as quantum mechanics and nonlinear optics. The Riccati-type expansion is recognized as an effective analytical technique for solving NLPDEs; however, it exhibits significant differences in complexity, applicability, and methodology. However, its complexity necessitates greater computational resources and a more sophisticated level of mathematical expertise from users. The extended fractional Riccati expansion method follows the following steps [38]:

Step 1: A nonlinear conformable time-fractional PDE in two independent variables x and t,

$$Q(\,D_t^\alpha u,\; D_x^\alpha u,\; D_t^{2\alpha} u,\; D_x^{2\alpha} u,\; \cdots) = 0, \tag{3}$$

where $D_t^\alpha u$ and $D_x^\alpha u$ are conformable derivatives of u and the function $u = u(x, t)$ is unknown, Q is a polynomial and various partial derivatives in u. The traveling wave transformation is presented as [38]:

$$u(x,\;t) = u(\xi),\; \xi = \xi(x, t). \tag{4}$$

Assigning the wave variable, the nonlinear fractional PDE Equation (3) is concentrated into the following fractional ODE for $u = u(\xi)$:

$$\widetilde{Q}\left(u,\; \omega^\alpha D_\xi^\alpha u,\; D_\xi^\alpha u,\; \omega^{2\alpha} D_\xi^{2\alpha} u,\; D_\xi^{2\alpha} u,\; \cdots\right) = 0. \tag{5}$$

Step 2: Suppose that $u(\xi)$ can be stated by a finite power series of $M(\xi)$, then,

$$u(\xi) = \frac{\sum_{i=0}^{n} a_i M^i(\xi)}{\sum_{i=0}^{n} b_i M^i(\xi)},\; a_n \neq 0 \text{ and } b_n \neq 0, \tag{6}$$

where a_i and b_i are constants that will be resolved later. n provides a positive integer and can be resolved by assuming the homogeneous balance of the highest-ordered linear term with the nonlinear term in Equation (5) and $M = M(\xi)$ satisfies the fractional extended Riccati equation is given below:

$$D_\xi^\alpha M(\xi) = \mu + \lambda M(\xi) + \nu M^2(\xi),\;\; 0 < \alpha \leq 1, \tag{7}$$

where μ represents the source or forcing term, λ represents the linear growth or damping coefficient, and ν represents nonlinear interaction strength. The Mittag–Leffler function of one parameter:

$$E_\alpha(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(1 + k\alpha)},\;\; (\alpha > 0), \tag{8}$$

we get the following solutions of Equation (6):

Case 1: If $\lambda^2 - 4\mu\nu > 0$ and $\lambda\nu \neq 0$, then,

$$M_1 = -\frac{1}{2\nu}\left[\lambda + \sqrt{\lambda^2 - 4\mu\nu}\tanh\left(0.5\sqrt{\lambda^2 - 4\mu\nu}\,\xi,\; \alpha\right)\right]. \tag{9}$$

$$M_2 = -\frac{1}{2\nu}\left[\lambda + \sqrt{\lambda^2 - 4\mu\nu}\coth\left(0.5\sqrt{\lambda^2 - 4\mu\nu}\,\xi,\; \alpha\right)\right]. \tag{10}$$

Case 2: If $\lambda^2 - 4\mu\nu < 0$ and $\lambda\nu \neq 0$, then,

$$M_3 = \frac{1}{2\nu}\left[-\lambda + \sqrt{4\mu\nu - \lambda^2}\tan\left(0.5\sqrt{4\mu\nu - \lambda^2}\,\xi,\; \alpha\right)\right]. \tag{11}$$

$$M_4 = -\frac{1}{2\nu}\left[\lambda + \sqrt{4\mu\nu - \lambda^2}\cot\left(0.5\sqrt{4\mu\nu - \lambda^2}\,\xi,\; \alpha\right)\right]. \tag{12}$$

Case 3: If $\lambda^2 - 4\mu\nu > 0$ and $\mu\nu \neq 0$, then,

$$M_5 = \frac{2\mu\cosh\left(0.5\sqrt{\lambda^2 - 4\mu\nu}\,\xi,\; \alpha\right)}{\sqrt{\lambda^2 - 4\mu\nu}\sinh\left(0.5\sqrt{\lambda^2 - 4\mu\nu}\,\xi,\; \alpha\right) - \lambda\cosh\left(0.5\sqrt{\lambda^2 - 4\mu\nu}\,\xi,\; \alpha\right)}. \tag{13}$$

$$M_6 = -\frac{2\mu \sinh\left(0.5\sqrt{\lambda^2 - 4\mu\nu}\,\xi,\ \alpha\right)}{\lambda \sinh\left(0.5\sqrt{\lambda^2 - 4\mu\nu}\,\xi,\ \alpha\right) - \sqrt{\lambda^2 - 4\mu\nu}\cosh\left(0.5\sqrt{\lambda^2 - 4\mu\nu}\,\xi,\ \alpha\right)}. \tag{14}$$

Case 4: If $\lambda^2 - 4\mu\nu < 0$ and $\mu\nu \neq 0$, then,

$$M_7 = -\frac{2\mu \cos\left(0.5\sqrt{4\mu\nu - \lambda^2}\,\xi,\ \alpha\right)}{\sqrt{4\mu\nu - \lambda^2}\sin\left(0.5\sqrt{4\mu\nu - \lambda^2}\,\xi,\ \alpha\right) + \lambda\cos\left(0.5\sqrt{4\mu\nu - \lambda^2}\,\xi,\ \alpha\right)}. \tag{15}$$

$$M_8 = -\frac{2\mu \sin\left(0.5\sqrt{4\mu\nu - \lambda^2}\,\xi,\ \alpha\right)}{\lambda \sin\left(0.5\sqrt{4\mu\nu - \lambda^2}\,\xi,\ \alpha\right) - \sqrt{4\mu\nu - \lambda^2}\cos\left(0.5\sqrt{4\mu\nu - \lambda^2}\,\xi,\ \alpha\right)}. \tag{16}$$

Case 5: If $\mu = 0$, $\lambda\nu \neq 0$, then,

$$M_9 = \frac{-\lambda r}{\nu[r + \cosh(\lambda\xi,\ \alpha) - \sinh(\lambda\xi,\ \alpha)]}. \tag{17}$$

$$M_{10} = \frac{-\lambda[\cosh(\lambda\xi,\ \alpha) + \sinh(\lambda\xi,\ \alpha)]}{\nu[r + \cosh(\lambda\xi,\ \alpha) + \sinh(\lambda\xi,\ \alpha)]}, \tag{18}$$

where r is arbitrary constant and the generalized hyperbolic and trigonometric functions are well-defined as:

$$\cosh(\xi,\ \alpha) = \frac{E_\alpha(\xi^\alpha) + E_\alpha(-\xi^\alpha)}{2},$$
$$\sinh(\xi,\ \alpha) = \frac{E_\alpha(\xi^\alpha) - E_\alpha(-\xi^\alpha)}{2}.$$

$$\cos(\xi,\ \alpha) = \frac{E_\alpha(i\xi^\alpha) + E_\alpha(-i\xi^\alpha)}{2},$$
$$\sin(\xi,\ \alpha) = \frac{E_\alpha(i\xi^\alpha) - E_\alpha(-i\xi^\alpha)}{2}.$$

$$\tanh(\xi,\ \alpha) = \frac{\sinh(\xi,\ \alpha)}{\cosh(\xi,\ \alpha)},\ \tan(\xi,\ \alpha) = \frac{\sin(\xi,\ \alpha)}{\cos(\xi,\ \alpha)}.$$

$$\coth(\xi,\ \alpha) = \frac{1}{\tanh(\xi,\ \alpha)},\ \cot(\xi,\ \alpha) = \frac{1}{\tan(\xi,\ \alpha)}.$$

Step 3: More precisely, we define the degree of $u(\xi)$ as $D(u(\xi)) = n$ which gives rise to the degree of other expression as follows:

$$D\left(\frac{d^q u}{d\xi^q}\right) = n + q,\ D\left(u^p\left(\frac{d^q u}{d\xi^q}\right)^s\right) = np + s(n + q).$$

Step 4: Demarking the value of the integer n, substituting Equation (6) with Equation (7) into Equation (5), plucking the terms with the same order of $M(\xi)$, and then, assigning each coefficient of $F(\xi)$ to zero which consigns a system of algebraic equations for the unknown parameters available in Equation (5). Solving this attained system by computational software Maple yields the values of $a_i's$ and $b_i's$.

Step 5: Inserting the values evaluated in Step 3 in Equation (6) along with the solutions of Equation (7) provides the traveling wave solutions to Equation (3).

4. Applications

While beta derivatives, such as the Atangana–Baleanu derivative, introduce modifications to address limitations in traditional fractional derivatives and provide a more generalized framework for modeling physical phenomena, conformable derivatives are defined as a limit-based approach that permits a continuous parameter that governs the order of the derivative. The KG Model Equation (1) in terms of conformable fractional derivative is:

$$D_t^{2\alpha}u - D_x^{2\alpha}u - \psi u - \varepsilon u^3 = 0, t > 0,\ 0 < \alpha \leq 1, \tag{19}$$

where $D_t^{2\alpha}u$ and $D_x^{2\alpha}u$ are satisfying conformable derivatives. The KG Model Equation (1) in terms of beta fractional derivative is:

$$ {}_0^M\mathscr{D}_t^{2\beta} u - {}_0^M\mathscr{D}_x^{2\beta} u - \psi u - \varepsilon u^3 = 0, t>0, 0<\beta \leq 1, \tag{20}$$

where ${}_0^M\mathscr{D}_t^{\beta} u$ and ${}_0^M\mathscr{D}_x^{\beta} u$ are satisfying the beta derivative and x represents spatial variable and t symbolizes time. This equation describes all spineless particles with positive, negative, and zero charge. This equation arises in nonlinear optics, nuclear physics, and geo physics. We introduce the fractional composite transformation as follows:

$$ u(x,\ t) = V(\xi). \tag{21}$$

The traveling wave parameter ξ is defined in two ways. The conformable fractional derivative, ξ has the following form:

$$ \xi = \frac{\tau x^\alpha}{\alpha} + \frac{\varphi t^\alpha}{\alpha}. \tag{22}$$

For the beta fractional derivative, it ξ has the following form:

$$ \xi = \tfrac{\tau}{\beta}\left(x + \frac{1}{\Gamma(\beta)}\right)^\beta + \tfrac{\varphi}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^\beta, \tag{23}$$

where τ and φ indicates wave number and speed of the traveling wave, respectively, that will be explored later. Wave propagation in nonlinear systems can be modeled using the beta derivative, a sort of fractional derivative, which frequently entails transformations that link the wave number and wave speed.

$$ (\tau^2 - \varphi^2)V'' + \psi V + \varepsilon V^3 = 0. \tag{24}$$

Balancing the highest-order derivative (V'') and the highest-order nonlinear term (V^3) yields $n = 1$. Then, the solution becomes:

$$ V(\xi) = \frac{a_0 + a_1 M(\xi)}{b_0 + b_1 M(\xi)}, a_n \neq 0 \text{ or } b_n \neq 0. \tag{25}$$

Equations (24) and (25), along with their necessary derivatives, together form a polynomial in $M(\xi)$. By collecting the coefficients of this polynomial and setting them to zero, we obtain:

$$ \begin{aligned} M(\xi)^0 :& - \mu b_0 b_1 \lambda \tau^2 a_0 + \lambda \mu \tau^2 a_1 b_0^2 + \mu b_0 b_1 \lambda \varphi^2 a_0 \\ & - \lambda \mu \varphi^2 a_1 b_0^2 + 2\mu^2 \tau^2 a_0 b_1^2 - 2\mu^2 b_0 b_1 \tau^2 a_1 \\ & - 2\mu^2 \varphi^2 a_0 b_1^2 + 2\mu^2 b_0 b_1 \varphi^2 a_1 + \psi a_0 b_0^2 + \varepsilon a_0^3 = 0. \end{aligned} $$

$$ \begin{aligned} M(\xi)^1 :& - b_0 b_1 a_0 \lambda^2 \tau^2 + a_1 b_0^2 \lambda^2 \tau^2 + b_0 b_1 a_0 \lambda^2 \varphi^2 \\ & - a_1 b_0^2 \lambda^2 \varphi^2 + 3\lambda \mu \tau^2 a_0 b_1^2 - 3b_0 b_1 \lambda \mu \tau^2 a_1 \\ & - 3\lambda \mu \varphi^2 a_0 b_1^2 + 3b_0 b_1 \lambda \mu \varphi^2 a_1 - 2b_0 b_1 a_0 \mu \nu \tau^2 \\ & + 2a_1 b_0^2 \mu \nu \tau^2 + 2b_0 b_1 a_0 \mu \nu \varphi^2 - 2a_1 b_0^2 \mu \nu \varphi^2 \\ & + 2b_0 b_1 a_0 \psi + a_1 b_0^2 \psi + 3a_1 \varepsilon a_0^2 = 0. \end{aligned} $$

$$ \begin{aligned} M(\xi)^2 :& \ a_0 b_1^2 \lambda^2 \tau^2 - b_0 b_1 a_1 \lambda^2 \tau^2 - a_0 b_1^2 \lambda^2 \varphi^2 \\ & + b_0 b_1 a_1 \lambda^2 \varphi^2 - 3\lambda \nu \tau^2 a_0 b_0 b_1 + 3a_1 \nu \lambda b_0^2 \tau^2 \\ & + 3\lambda \nu \varphi^2 a_0 b_0 b_1 - 3a_1 \nu \lambda b_0^2 \varphi^2 + 2a_0 b_1^2 \mu \nu \tau^2 \\ & - 2b_0 b_1 a_1 \mu \nu \tau^2 - 2a_0 b_1^2 \mu \nu \varphi^2 + 2b_0 b_1 a_1 \mu \nu \varphi^2 \\ & + a_0 b_1^2 \psi + 2b_0 b_1 a_1 \psi + 3\varepsilon a_0 a_1^2 = 0. \end{aligned} $$

$$ \begin{aligned} M(\xi)^3 :& \ \lambda \nu \tau^2 a_0 b_1^2 - \lambda \nu \tau^2 a_1 b_0 b_1 - \lambda \nu \varphi^2 a_0 b_1^2 \\ & + \lambda \nu \varphi^2 a_1 b_0 b_1 - 2\nu^2 \tau^2 a_0 b_0 b_1 + 2\nu^2 \tau^2 a_1 b_0^2 \\ & + 2\nu^2 \varphi^2 a_0 b_0 b_1 - 2\nu^2 \varphi^2 a_1 b_0^2 + \psi a_1 b_1^2 + \varepsilon a_1^3 = 0. \end{aligned} $$

Solving these equations using the computational software Maple yields result Sets 1–3.

Set 1

$$ \begin{aligned} & a_0 = \pm \frac{i\sqrt{\psi}}{\sqrt{\varepsilon}} \times \frac{b_1 \sqrt{(\lambda^2 - 4\mu\nu)}}{2\nu}, a_1 = 0, b_0 = \frac{\lambda b_1}{2\nu}, \\ & b_1 = b_1, \varphi = \pm \frac{\sqrt{\tau^2(\lambda^2 - 4\mu\nu) - 2\psi}}{\sqrt{\lambda^2 - 4\mu\nu}}. \end{aligned} $$

Set 2

$$ \begin{aligned} & a_0 = \pm \frac{i\sqrt{\psi}}{\sqrt{\varepsilon}} \times \frac{\lambda b_0}{\sqrt{(\lambda^2 - 4\mu\nu)}}, a_1 = \pm \frac{i\sqrt{\psi}}{\sqrt{\varepsilon}} \times \frac{2b_0 \nu}{\sqrt{(\lambda^2 - 4\mu\nu)}}, \\ & b_0 = b_0, b_1 = 0, \varphi = \pm \frac{\sqrt{\tau^2(\lambda)^2(-4\mu\nu) - 2\psi}}{\sqrt{\lambda^2 - 4\mu\nu}}. \end{aligned} $$

Set 3

$$ \begin{aligned} & a_0 = \pm \frac{i\sqrt{\psi}}{\sqrt{\varepsilon(\lambda^2 - 4\mu\nu)}}(\lambda b_0 - 2\mu b_1), \\ & a_1 = \pm \frac{i\sqrt{\psi}}{\sqrt{\varepsilon(\lambda^2 = 4\mu\nu)}}(\lambda b_1 - 2\nu b_0), \\ & b_0 = b_0, b_1 = b_1, \varphi = \pm \frac{\sqrt{\tau^2(\lambda^2 - 4\mu\nu) - 2\psi}}{\sqrt{\lambda^2 - 4\mu\nu}}. \end{aligned} $$

Substituting Set 1 to Set 3 with Equation (25) make available the following wave solutions:

Solution family 1: The general expression of the solution due to Set 1 is:

$$ u(x,\ t) = \pm \frac{i\sqrt{\psi}}{\sqrt{\varepsilon}} \times \frac{\sqrt{(\lambda^2 - 4\mu\nu)}}{(\lambda + 2\nu M(\xi))}, \tag{26}$$

where, $\xi(x,t) = \frac{\tau x^\alpha}{\alpha} + \frac{\varphi t^\alpha}{\alpha}$, or $\xi = \frac{\tau}{\beta}\left(x + \frac{1}{\Gamma(\beta)}\right)^\beta + \frac{\varphi}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^\beta$ and $\varphi = \pm \frac{\sqrt{\tau^2(\lambda^2 - 4\mu\nu) - 2\psi}}{\sqrt{\lambda^2 - 4\mu\nu}}$.

$\lambda^2 - 4\mu\nu > 0$ and $\lambda\nu \neq 0$,

$$u_{1,2}(x,\ t) = \pm \frac{i\sqrt{\psi}}{\sqrt{\varepsilon}} \times \frac{1}{\tanh\left(0.5\sqrt{\lambda^2 - 4\mu\nu}\,\xi,\ \alpha\right)}, \tag{27}$$

$$u_{3,4}(x,\ t) = \pm \frac{i\sqrt{\psi}}{\sqrt{\varepsilon}} \times \frac{1}{\coth\left(0.5\sqrt{\lambda^2 - 4\mu\nu}\,\xi,\ \alpha\right)}, \tag{28}$$

When $\lambda^2 - 4\mu\nu < 0$ and $\lambda\nu \neq 0$,

$$u_{5,6}(x,\ t) = \pm \frac{\sqrt{\psi}}{\sqrt{\varepsilon}} \times \frac{1}{\tan\left(0.5\sqrt{4\mu\nu - \lambda^2}\,\xi,\ \alpha\right)}, \tag{29}$$

$$u_{7,8}(x,\ t) = \pm \frac{\sqrt{\psi}}{\sqrt{\varepsilon}} \times \frac{1}{\cot\left(0.5\sqrt{4\mu\nu - \lambda^2}\,\xi,\ \alpha\right)}. \tag{30}$$

Due to $\lambda^2 - 4\mu\nu > 0$ and $\mu\nu \neq 0$,

$$u_{9,10}(x,\ t) = \pm \frac{i\sqrt{\psi}}{\sqrt{\varepsilon}} \times \frac{\sqrt{\lambda^2 - 4\mu\nu}\sinh\left(0.5\sqrt{\lambda^2 - 4\mu\nu}\,\xi,\ \alpha\right) - \lambda\cosh\left(0.5\sqrt{\lambda^2 - 4\mu\nu}\,\xi,\ \alpha\right)}{\lambda\sinh\left(0.5\sqrt{\lambda^2 - 4\mu\nu}\,\xi,\ \alpha\right) - \sqrt{\lambda^2 - 4\mu\nu}\cosh\left(0.5\sqrt{\lambda^2 - 4\mu\nu}\,\xi,\ \alpha\right)}, \tag{31}$$

$$u_{11,12}(x,\ t) = \pm \frac{i\sqrt{\psi}}{\sqrt{\varepsilon}} \times \frac{\lambda\sinh\left(0.5\sqrt{\lambda^2 - 4\mu\nu}\,\xi,\ \alpha\right) - \sqrt{\lambda^2 - 4\mu\nu}\cosh\left(0.5\sqrt{\lambda^2 - 4\mu\nu}\,\xi,\ \alpha\right)}{\sqrt{\lambda^2 - 4\mu\nu}\sinh\left(0.5\sqrt{\lambda^2 - 4\mu\nu}\,\xi,\ \alpha\right) - \lambda\cosh\left(0.5\sqrt{\lambda^2 - 4\mu\nu}\,\xi,\ \alpha\right)}. \tag{32}$$

While $\lambda^2 - 4\mu\nu < 0$ and $\mu\nu \neq 0$,

$$u_{13,14}(x,\ t) = \pm \frac{\sqrt{\psi}}{\sqrt{\varepsilon}} \times \frac{\sqrt{4\mu\nu - \lambda^2}\sin\left(0.5\sqrt{4\mu\nu - \lambda^2}\,\xi,\ \alpha\right) + \lambda\cos\left(0.5\sqrt{4\mu\nu - \lambda^2}\,\xi,\ \alpha\right)}{\lambda\sin\left(0.5\sqrt{4\mu\nu - \lambda^2}\,\xi,\ \alpha\right) + \sqrt{4\mu\nu - \lambda^2}\cos\left(0.5\sqrt{4\mu\nu - \lambda^2}\,\xi,\ \alpha\right)}, \tag{33}$$

$$u_{15,16}(x,\ t) = \pm \frac{\sqrt{\psi}}{\sqrt{\varepsilon}} \times \frac{\lambda\sin\left(0.5\sqrt{4\mu\nu - \lambda^2}\,\xi,\ \alpha\right) - \sqrt{4\mu\nu - \lambda^2}\cos\left(0.5\sqrt{4\mu\nu - \lambda^2}\,\xi,\ \alpha\right)}{\sqrt{4\mu\nu - \lambda^2}\sin\left(0.5\sqrt{4\mu\nu - \lambda^2}\,\xi,\ \alpha\right) + \lambda\cos\left(0.5\sqrt{4\mu\nu - \lambda^2}\,\xi,\ \alpha\right)}. \tag{34}$$

If $\mu = 0$ and $\lambda\nu \neq 0$,

$$u_{17,18}(x,\ t) = \pm \frac{i\sqrt{\psi}}{\sqrt{\varepsilon}} \times \frac{r + \cosh(\lambda\xi,\ \alpha) - \sinh(\lambda\xi,\ \alpha)}{\cosh(\lambda\xi,\ \alpha) - \sinh(\lambda\xi,\ \alpha) - r}, \tag{35}$$

$$u_{19,20}(x,\ t) = \pm \frac{i\sqrt{\psi}}{\sqrt{\varepsilon}} \times \frac{r + \cosh(\lambda\xi,\ \alpha) + \sinh(\lambda\xi,\ \alpha)}{r - \cosh(\lambda\xi,\ \alpha) - \sinh(\lambda\xi,\ \alpha)}, \tag{36}$$

where, in the sense of comfortable derivative $\xi(x,t) = \frac{\tau x^\alpha}{\alpha} + \frac{\varphi t^\alpha}{\alpha}$, beta derivative $\xi = \frac{\tau}{\beta}(x + \frac{1}{\Gamma(\beta)})^\beta + \frac{\varphi}{\beta}(t + \frac{1}{\Gamma(\beta)})^\beta$, and $\varphi = \pm \frac{\sqrt{\tau^2(\lambda^2 - 4\mu\nu) - 2\psi}}{\sqrt{\lambda^2 - 4\mu\nu}}$.

Solution family 2: The general expression of the solution due to Set 2 is:

$$u(x,\ t) = \pm \frac{i\sqrt{\psi}}{\sqrt{\varepsilon}} \times \frac{\lambda + 2\nu M(\xi)}{\sqrt{\lambda^2 - 4\mu\nu}}, \tag{37}$$

where, $\xi(x,t) = \frac{\tau x^\alpha}{\alpha} + \frac{\varphi t^\alpha}{\alpha}$ or $\xi = \frac{\tau}{\beta}\left(x + \frac{1}{\Gamma(\beta)}\right)^\beta + \frac{\varphi}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^\beta$ and $\varphi = \pm \frac{\sqrt{\tau^2(\lambda^2 - 4\mu\nu) - 2\psi}}{\sqrt{\lambda^2 - 4\mu\nu}}$.

When $\lambda^2 - 4\mu\nu > 0$ and $\lambda\nu \neq 0$,

$$u_{21,22}(x,\ t) = \pm \frac{i\sqrt{\psi}}{\sqrt{\varepsilon}} \times \tanh\left(0.5\sqrt{\lambda^2 - 4\mu\nu}\,\xi,\ \alpha\right), \tag{38}$$

$$u_{23,24}(x,\ t) = \pm \frac{i\sqrt{\psi}}{\sqrt{\varepsilon}} \times \coth\left(0.5\sqrt{\lambda^2 - 4\mu\nu}\,\xi,\ \alpha\right). \tag{39}$$

When $\lambda^2 - 4\mu\nu < 0$ and $\lambda\nu \neq 0$,

$$u_{25,26}(x,\ t) = \pm \frac{\sqrt{\psi}}{\sqrt{\varepsilon}} \times \tan\left(0.5\sqrt{4\mu\nu - \lambda^2}\,\xi,\ \alpha\right), \quad (40)$$

$$u_{27,28}(x,\ t) = \pm \frac{\sqrt{\psi}}{\sqrt{\varepsilon}} \times \cot\left(0.5\sqrt{4\mu\nu - \lambda^2}\,\xi,\ \alpha\right). \quad (41)$$

Due to $\lambda^2 - 4\mu\nu > 0$ and $\mu\nu \neq 0$,

$$u_{29,30}(x,\ t) = \pm \frac{i\sqrt{\psi}}{\sqrt{\varepsilon}} \times \frac{\lambda \sinh\left(0.5\sqrt{\lambda^2 - 4\mu\nu}\,\xi,\ \alpha\right) - \sqrt{\lambda^2 - 4\mu\nu}\cosh\left(0.5\sqrt{\lambda^2 - 4\mu\nu}\,\xi,\ \alpha\right)}{\sqrt{\lambda^2 - 4\mu\nu}\sinh\left(0.5\sqrt{\lambda^2 - 4\mu\nu}\,\xi,\ \alpha\right) - \lambda\cosh\left(0.5\sqrt{\lambda^2 - 4\mu\nu}\,\xi,\ \alpha\right)}, \quad (42)$$

$$u_{31,32}(x,\ t) = \pm \frac{i\sqrt{\psi}}{\sqrt{\varepsilon}} \times \frac{\sqrt{\lambda^2 - 4\mu\nu}\sinh\left(0.5\sqrt{\lambda^2 - 4\mu\nu}\,\xi,\ \alpha\right) - \lambda\cosh\left(0.5\sqrt{\lambda^2 - 4\mu\nu}\,\xi,\ \alpha\right)}{\lambda \sinh\left(0.5\sqrt{\lambda^2 - 4\mu\nu}\,\xi,\ \alpha\right) - \sqrt{\lambda^2 - 4\mu\nu}\cosh\left(0.5\sqrt{\lambda^2 - 4\mu\nu}\,\xi,\ \alpha\right)}. \quad (43)$$

While $\lambda^2 - 4\mu\nu < 0$ and $\mu\nu \neq 0$,

$$u_{33,34}(x,\ t) = \pm \frac{\sqrt{\psi}}{\sqrt{\varepsilon}} \times \frac{\lambda \sin\left(0.5\sqrt{4\mu\nu - \lambda^2}\,\xi,\ \alpha\right) - \sqrt{4\mu\nu - \lambda^2}\cos\left(0.5\sqrt{4\mu\nu - \lambda^2}\,\xi,\ \alpha\right)}{\sqrt{4\mu\nu - \lambda^2}\sin\left(0.5\sqrt{4\mu\nu - \lambda^2}\,\xi,\ \alpha\right) + \lambda\cos\left(0.5\sqrt{4\mu\nu - \lambda^2}\,\xi,\ \alpha\right)}, \quad (44)$$

$$u_{35,36}(x,\ t) = \pm \frac{\sqrt{\psi}}{\sqrt{\varepsilon}} \times \frac{\lambda \cos\left(0.5\sqrt{4\mu\nu - \lambda^2}\,\xi,\ \alpha\right) - \sqrt{4\mu\nu - \lambda^2}\sin\left(0.5\sqrt{4\mu\nu - \lambda^2}\,\xi,\ \alpha\right)}{\lambda\sin\left(0.5\sqrt{4\mu\nu - \lambda^2}\,\xi,\ \alpha\right) - \sqrt{4\mu\nu - \lambda^2}\cos\left(0.5\sqrt{4\mu\nu - \lambda^2}\,\xi,\ \alpha\right)}. \quad (45)$$

If $\mu = 0$ and $\lambda\nu \neq 0$,

$$u_{37,38}(x,\ t) = \pm \frac{i\sqrt{\psi}}{\sqrt{\varepsilon}} \times \frac{\cosh(\lambda\xi,\ \alpha) - \sinh(\lambda\xi,\ \alpha) - r}{r + \cosh(\lambda\xi,\ \alpha) - \sinh(\lambda\xi,\ \alpha)}, \quad (46)$$

$$u_{39,40}(x,\ t) = \pm \frac{i\sqrt{\psi}}{\sqrt{\varepsilon}} \times \frac{r - \cosh(\lambda\xi,\ \alpha) - \sinh(\lambda\xi,\ \alpha)}{r + \cosh(\lambda\xi,\ \alpha) + \sinh(\lambda\xi,\ \alpha)}, \quad (47)$$

where, $\xi(x,t) = \frac{\tau x^\alpha}{\alpha} + \frac{\varphi t^\alpha}{\alpha}$ or $\xi = \frac{\tau}{\beta}\left(x + \frac{1}{\Gamma(\beta)}\right)^\beta + \frac{\varphi}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^\beta$ and $\varphi = \pm\frac{\sqrt{\tau^2(\lambda^2 - 4\mu\nu) - 2\psi}}{\sqrt{\lambda^2 - 4\mu\nu}}$.

Solution family 3: The general expression of the solution due to Set 3 are:

$$u(x,\ t) = \pm \frac{i\sqrt{\psi}}{\sqrt{\varepsilon}} \times \frac{(\lambda b_0 - 2\mu b_1) + (\lambda b_1 - 2\nu b_0)M}{\sqrt{\lambda^2 - 4\mu\nu}\,(b_0 + b_1 M)}, \quad (48)$$

where, $\xi(x,t) = \frac{\tau x^\alpha}{\alpha} + \frac{\varphi t^\alpha}{\alpha}$ or $\xi = \frac{\tau}{\beta}\left(x + \frac{1}{\Gamma(\beta)}\right)^\beta + \frac{\varphi}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^\beta$ and $\varphi = \pm\frac{\sqrt{\tau^2(\lambda^2 - 4\mu\nu) - 2\psi}}{\sqrt{\lambda^2 - 4\mu\nu}}$.

If $\lambda^2 - 4\mu\nu > 0$ and $\lambda\nu \neq 0$,

$$u_{41,42}(x,\ t) = \pm \frac{i\sqrt{\psi}}{\sqrt{\varepsilon}\sqrt{\lambda^2 - 4\mu\nu}} \times \frac{2\nu(\lambda b_0 - 2\mu b_1) - (\lambda b_1 - 2\nu b_0)\left(\lambda + \sqrt{\lambda^2 - 4\mu\nu}\tanh\left(0.5\sqrt{\lambda^2 - 4\mu\nu}\,\xi,\ \alpha\right)\right)}{2\nu b_0 - b_1\left(\lambda + \sqrt{\lambda^2 - 4\mu\nu}\tanh\left(0.5\sqrt{\lambda^2 - 4\mu\nu}\,\xi,\ \alpha\right)\right)}, \quad (49)$$

$$u_{43,44}(x,\ t)=\pm\frac{i\sqrt{\psi}}{\sqrt{\varepsilon}\sqrt{\lambda^2-4\mu\nu}}\times\frac{2\nu(\lambda b_0-2\mu b_1)-(\lambda b_1-2\nu b_0)\left(\lambda+\sqrt{\lambda^2-4\mu\nu}\coth\left(0.5\sqrt{\lambda^2-4\mu\nu}\,\xi,\ \alpha\right)\right)}{2\nu b_0-b_1\left(\lambda+\sqrt{\lambda^2-4\mu\nu}\coth\left(0.5\sqrt{\lambda^2-4\mu\nu}\,\xi,\ \alpha\right)\right)}. \tag{50}$$

When $\lambda^2-4\mu\nu<0$ and $\lambda\nu\neq 0$,

$$u_{45,46}(x,\ t)=\pm\frac{\sqrt{\psi}}{\sqrt{\varepsilon}\sqrt{4\mu\nu-\lambda^2}}\times\frac{2\nu(\lambda b_0-2\mu b_1)+(\lambda b_1-2\nu b_0)\left(-\lambda+\sqrt{4\mu\nu-\lambda^2}\tan\left(0.5\sqrt{4\mu\nu-\lambda^2}\,\xi,\ \alpha\right)\right)}{2\nu b_0+b_1\left(-\lambda+\sqrt{4\mu\nu-\lambda^2}\tan\left(0.5\sqrt{4\mu\nu-\lambda^2}\,\xi,\ \alpha\right)\right)}, \tag{51}$$

$$u_{47,48}(x,\ t)=\pm\frac{\sqrt{\psi}}{\sqrt{\varepsilon}\sqrt{4\mu\nu-\lambda^2}}\times\frac{2\nu(\lambda b_0-2\mu b_1)-(\lambda b_1-2\nu b_0)\left(\lambda+\sqrt{4\mu\nu-\lambda^2}\cot\left(0.5\sqrt{4\mu\nu-\lambda^2}\,\xi,\ \alpha\right)\right)}{2\nu b_0-b_1\left(\lambda+\sqrt{4\mu\nu-\lambda^2}\cot\left(0.5\sqrt{4\mu\nu-\lambda^2}\,\xi,\ \alpha\right)\right)}. \tag{52}$$

Due to $\lambda^2-4\mu\nu>0$ and $\mu\nu\neq 0$,

$$\begin{aligned}u_{49,50}(x,\ t)&=\pm\frac{i\sqrt{\psi}}{\sqrt{\varepsilon}\sqrt{\lambda^2-4\mu\nu}}\\&\times\frac{(\lambda b_0-2\mu b_1)\left(\sqrt{\lambda^2-4\mu\nu}\sinh\left(0.5\sqrt{\lambda^2-4\mu\nu}\,\xi,\ \alpha\right)-\lambda\cosh\left(0.5\sqrt{\lambda^2-4\mu\nu}\,\xi,\ \alpha\right)\right)+(\lambda b_1-2\nu b_0)\left(2\mu\cosh\left(0.5\sqrt{\lambda^2-4\mu\nu}\,\xi,\ \alpha\right)\right)}{b_0\left(\sqrt{\lambda^2-4\mu\nu}\sinh\left(0.5\sqrt{\lambda^2-4\mu\nu}\,\xi,\ \alpha\right)-\lambda\cosh\left(0.5\sqrt{\lambda^2-4\mu\nu}\,\xi,\ \alpha\right)\right)+2\mu b_1\cosh\left(0.5\sqrt{\lambda^2-4\mu\nu}\,\xi,\ \alpha\right)},\end{aligned} \tag{53}$$

$$\begin{aligned}u_{51,52}(x,\ t)&=\pm\frac{i\sqrt{\psi}}{\sqrt{\varepsilon}\sqrt{\lambda^2-4\mu\nu}}\\&\times\frac{(\lambda b_0-2\mu b_1)\left(\lambda\sinh\left(0.5\sqrt{\lambda^2-4\mu\nu}\,\xi,\ \alpha\right)-\sqrt{\lambda^2-4\mu\nu}\cosh\left(0.5\sqrt{\lambda^2-4\mu\nu}\,\xi,\ \alpha\right)\right)-(\lambda b_1-2\nu b_0)\left(2\mu\sinh\left(0.5\sqrt{\lambda^2-4\mu\nu}\,\xi,\ \alpha\right)\right)}{b_0\left(\lambda\sinh\left(0.5\sqrt{\lambda^2-4\mu\nu}\,\xi,\ \alpha\right)-\sqrt{\lambda^2-4\mu\nu}\cosh\left(0.5\sqrt{\lambda^2-4\mu\nu}\,\xi,\ \alpha\right)\right)-2\mu b_1\sinh\left(0.5\sqrt{\lambda^2-4\mu\nu}\,\xi,\ \alpha\right)}.\end{aligned} \tag{54}$$

While $\lambda^2-4\mu\nu<0$ and $\mu\nu\neq 0$,

$$\begin{aligned}u_{53,54}(x,\ t)&=\pm\frac{\sqrt{\psi}}{\sqrt{\varepsilon}\sqrt{4\mu\nu-\lambda^2}}\\&\times\frac{(\lambda b_0-2\mu b_1)\left(\sqrt{4\mu\nu-\lambda^2}\sin\left(0.5\sqrt{4\mu\nu-\lambda^2}\,\xi,\ \alpha\right)+\lambda\cos\left(0.5\sqrt{4\mu\nu-\lambda^2}\,\xi,\ \alpha\right)\right)-(\lambda b_1-2\nu b_0)\left(2\mu\cos\left(0.5\sqrt{4\mu\nu-\lambda^2}\,\xi,\ \alpha\right)\right)}{b_0\left(\sqrt{4\mu\nu-\lambda^2}\sin\left(0.5\sqrt{4\mu\nu-\lambda^2}\,\xi,\ \alpha\right)+\lambda\cos\left(0.5\sqrt{4\mu\nu-\lambda^2}\,\xi,\ \alpha\right)\right)-2\mu b_1\cos\left(0.5\sqrt{4\mu\nu-\lambda^2}\,\xi,\ \alpha\right)},\end{aligned} \tag{55}$$

$$\begin{aligned}u_{55,56}(x,\ t)&=\pm\frac{\sqrt{\psi}}{\sqrt{\varepsilon}\sqrt{4\mu\nu-\lambda^2}}\\&\times\frac{(\lambda b_0-2\mu b_1)\left(\lambda\sin\left(0.5\sqrt{4\mu\nu-\lambda^2}\,\xi,\ \alpha\right)-\sqrt{4\mu\nu-\lambda^2}\cos\left(0.5\sqrt{4\mu\nu-\lambda^2}\,\xi,\ \alpha\right)\right)-(\lambda b_1-2\nu b_0)\left(2\mu\sin\left(0.5\sqrt{4\mu\nu-\lambda^2}\,\xi,\ \alpha\right)\right)}{b_0\left(\lambda\sin\left(0.5\sqrt{4\mu\nu-\lambda^2}\,\xi,\ \alpha\right)-\sqrt{4\mu\nu-\lambda^2}\cos\left(0.5\sqrt{4\mu\nu-\lambda^2}\,\xi,\ \alpha\right)\right)-2\mu b_1\sin\left(0.5\sqrt{4\mu\nu-\lambda^2}\,\xi,\ \alpha\right)}.\end{aligned} \tag{56}$$

If $\mu = 0$ and $\lambda\nu \neq 0$,

$$u_{57,58}(x,\ t) = \pm \frac{i\sqrt{\psi}}{\sqrt{\varepsilon}} \times \frac{\nu b_0(r + \cosh(\lambda\xi,\ \alpha) - \sinh(\lambda\xi,\ \alpha)) - r(\lambda b_1 - 2\nu b_0)}{b_0\nu(r + \cosh(\lambda\xi,\ \alpha) - \sinh(\lambda\xi,\ \alpha)) - b_1\lambda r}, \quad (57)$$

$$u_{59,60}(x,\ t) = \pm \frac{i\sqrt{\psi}}{\sqrt{\varepsilon}} \times \frac{\nu b_0(r + \cosh(\lambda\xi,\ \alpha) + \sinh(\lambda\xi,\ \alpha)) - (\lambda b_1 - 2\nu b_0)(\cosh(\lambda\xi,\ \alpha) + \sinh(\lambda\xi,\ \alpha))}{b_0\nu(r + \cosh(\lambda\xi,\ \alpha) + \sinh(\lambda\xi,\ \alpha)) - b_1\lambda(\cosh)(\lambda\xi,\ \alpha) + \sinh(\lambda\xi,\ \alpha)}, \quad (58)$$

$$\text{where, } \xi(x,t) = \frac{\tau x^\alpha}{\alpha} + \frac{\varphi t^\alpha}{\alpha} \text{ or } \xi = \frac{\tau}{\beta}\left(x + \frac{1}{\Gamma(\beta)}\right)^\beta + \frac{\varphi}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^\beta \text{ and } \varphi = \pm \frac{\sqrt{\tau^2(\lambda^2 - 4\mu\nu) - 2\psi}}{\sqrt{\lambda^2 - 4\mu\nu}}.$$

5. Results and Discussions

In this section, we employ the extended fractional Riccati expansion method, incorporating the conformable fractional derivative and beta fractional derivative approaches, to investigate whether the explicit and analytic traveling wave solutions of the KG equation are physically feasible. For the particular parameter's values in the solution that was obtained, the impact of the time-fractional derivative is recorded for the standards of the fractional parameters $\alpha, \beta = 0.1$, 0.5, and 0.9 with 3D graphical and corresponding density plots.

Figure 1a: The solution u_1 illustrates the breather wave shape in Figure 1a, characterized by parameters $\tau = 0.05$, $\psi = 1, \varepsilon = 1, \lambda = 3, \mu = 2, \nu = 1$, and $\alpha = 0.89$, utilizing the conformable derivative approach. The graph spans the domain $-3.5 \leq x \leq 3.5$ and $0 \leq t \leq 7$, highlighting the soliton's behavior over space and time. Figure 1b demonstrates the impact of varying α on the soliton shape, holding other parameters constant. It explores how changes in α influence the soliton's characteristics within the conformable derivative framework.

In Figure 1c, the influence of ε on the soliton's shape is depicted, while maintaining the other parameters constant. This graph illustrates how alterations in ε affect the behavior of the soliton under the conformable derivative setting. Figure 1d showcases the shape variation from using parameter α to β, providing insights into the transformation of the soliton under different conditions. The figure presents the Breather wave shape under consideration. Figure 1e: The impact of β on the soliton's shape is examined in Figure 1e, highlighting how adjustments in β influence the soliton's characteristics within the beta derivative framework, while other parameters remain constant. Figure 1f illustrates the influence of ε on the soliton's behavior under the beta derivative setting, maintaining other parameters constant. This figure elucidates how changes in ε affect the soliton's shape within this derivative framework.

In Figure 2, interaction between breather wave and king wave soliton solution illustrates from the solution $\mathrm{Re}(u_3(x, 0, 0, t))$ for the value of parameters $\tau = 0.05, \psi = 1, \varepsilon = 1, \lambda = 3, \mu = 2, \nu = 1,\ \alpha, \beta = 0.89$ in Figure 2a,d where the standards of the fractional parameters $\alpha, \beta = 0.1,\ 0.5,\ 0.9$ in Figure 2b,e, respectively. Also, we show the effect of $\varepsilon = 0.1,\ 0.5,\ 0.9$ in terms of conformable and beta derivative sense in Figure 2c,f.

Figure 3 reveals the solution for $\mathrm{Re}(u_5(x, 0, 0, t))$ for $\tau = 0.05,\ \psi = 1,\ \varepsilon = 1,\ \lambda = 1,\ \mu = 2,\ \nu = 1$. Figure 3a,d represents the bright bell shape and dark bell shape respectively, which validates the effect of alteration of the framework from conformable fractional derivative to beta fractional derivative assuming for $\alpha = 0.8$ and $\beta = 0.89$. Figure 3b,e illustrates the change in the shape of the soliton for $\alpha, \beta = 0.1, 0.5, 0.9$ remaining other parameters the same. Figure 3c,f detects the influence of ε beneath conformable fractional derivative and beta fractional derivative for $\varepsilon = 0.1$ and $\varepsilon = 0.1,\ 0.5,\ 0.9$, respectively.

Graphical representation of $\mathrm{Re}(u_7(x, 0, 0, t))$ characterized in Figure 4 for the values of the parameters $\tau = 0.05,\ \psi = 1,\ \varepsilon = 1,\ \lambda = 1,\ \mu = 2,\ \nu = 1$ demonstrating change under two context namely conformable fractional derivative and beta fractional derivative in Figure 4a,d for $\alpha = 0.5$ and $\beta = 0.89$, respectively, which represents the singular soliton wave. Figure 4b,e highlights the change in shape for different values of parameters for $\alpha = 0.1,\ 0.5,\ 0.9$ and $\beta = 0.1,\ 0.5,\ 0.9$. Figure 4c,f elucidates the influence ε on the shape of the soliton in a diverse derivative sense for $\varepsilon = 0.1$ and $\varepsilon = 0.1,\ 0.5,\ 0.9$, respectively.

Figure 5 is the pictorial depiction of the solution $(u_9(x, 0, 0, t))$. The periodic wave in Figure 5a,d is the demonstration of the variation in behavior of the soliton for different derivative senses for $\alpha = 0.7$ and $\beta = 0.89$, respectively, considering $\tau = 0.05,\ \psi = 1,\ \varepsilon = 1,\ \lambda = 3,\ \mu = 2,\ \nu = 1$. Figure 5b,e provides the characterization of the behavior of the soliton for $\alpha, \beta = 0.1,\ 0.5,\ 0.9$. Depiction of the impact of variation in the values of ε represented in Figure 5c,f for $\varepsilon = 0.1$ and $\varepsilon = 0.1,\ 0.5,\ 0.9$.

The periodic wave solution outlined in Figure 6 is the illustration of the solution $(u_{11}(x, 0, 0, t))$. Figure 6a depicts the characterization of the shape of the soliton under conformable fractional derivative for $\alpha = 0.7$, whereas Figure 6d depicts under beta fractional derivative for $\beta = 0.89$ considering other parameters as $\tau = 0.05,\ \psi = 1,\ \varepsilon = 1,\ \lambda = 3,\ \mu = 2,\ \nu = 1$. Figure 6b,e highlights the change of shape for $\alpha, \beta = 0.1$,

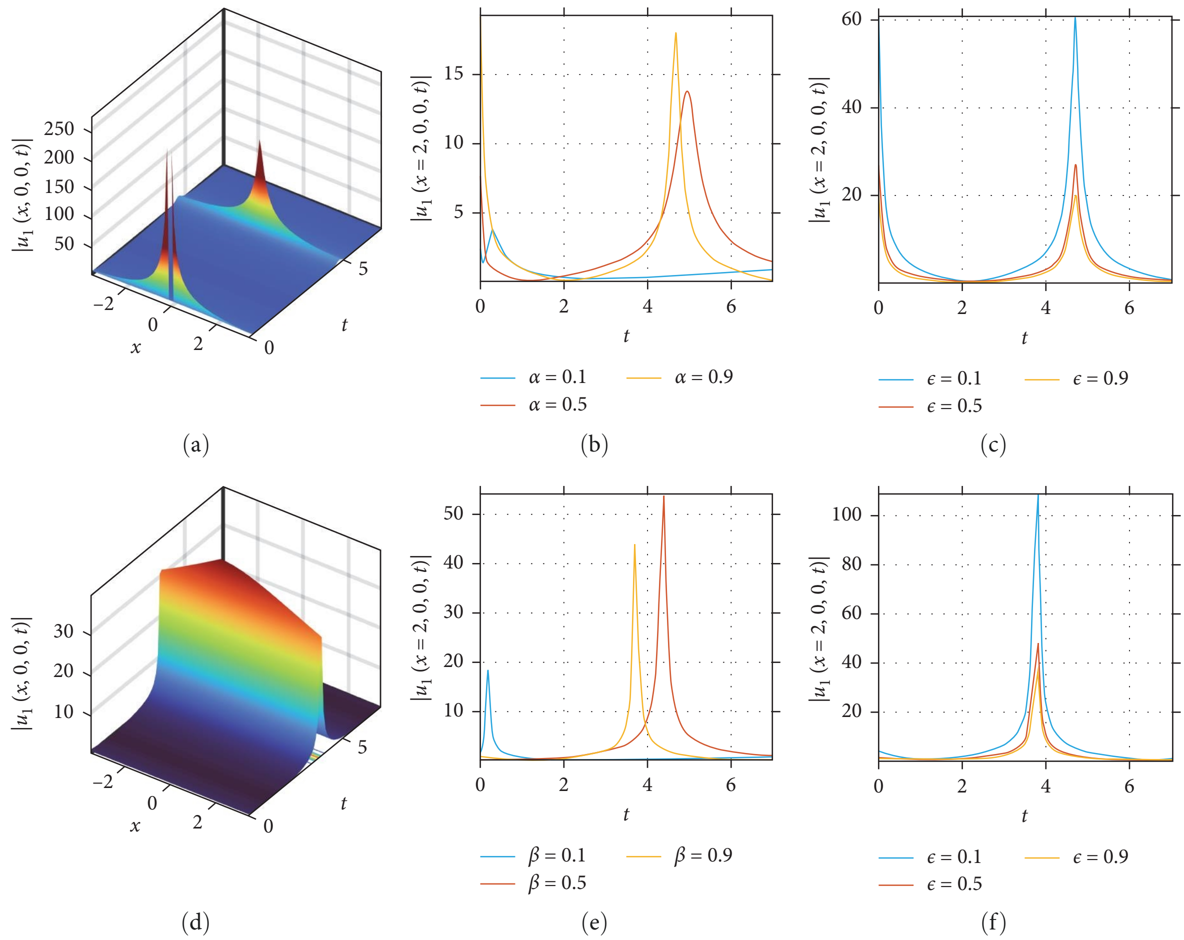

FIGURE 1: The solution $|u_1|$ for $\tau = 0.05, \psi = 1, \varepsilon = 1, \lambda = 3, \mu = 2, \nu = 1$, (a) $\alpha = 0.89$, (b) $\alpha = 0.1, 0.5, 0.9$, (c) $\varepsilon = 0.1, 0.5, 0.9$, (d) $\beta = 0.89$, (e) $\beta = 0.1, 0.5, 0.9$, and (f) $\varepsilon = 0.1, 0.5, 0.9$.

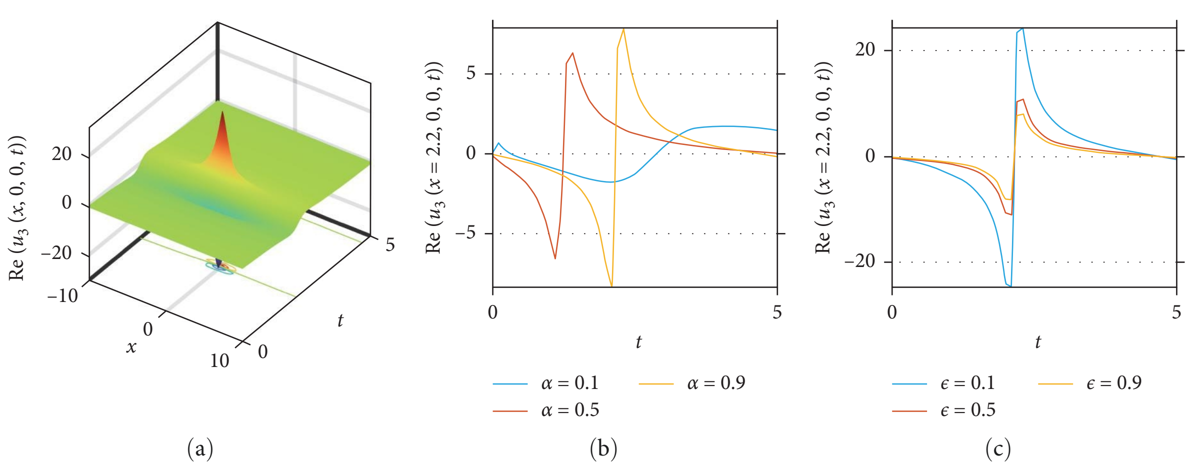

FIGURE 2: Continued.

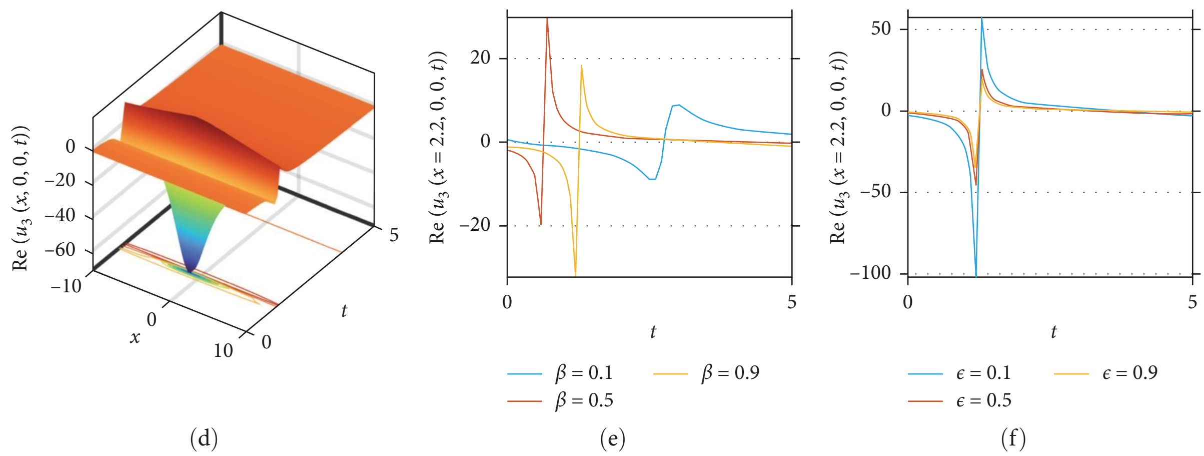

Figure 2: The solution $\mathrm{Re}(u_3)$ for $\tau = 0.05, \psi = 1, \varepsilon = 1, \lambda = 3, \mu = 2, \nu = 1,$ (a) $\alpha = 0.89$, (b) $\alpha = 0.1, 0.5, 0.9$, (c) $\varepsilon = 0.1, 0.5, 0.9$, (d) $\beta = 0.89$, (e) $\beta = 0.1, 0.5, 0.9$, and (f) $\varepsilon = 0.1, 0.5, 0.9$.

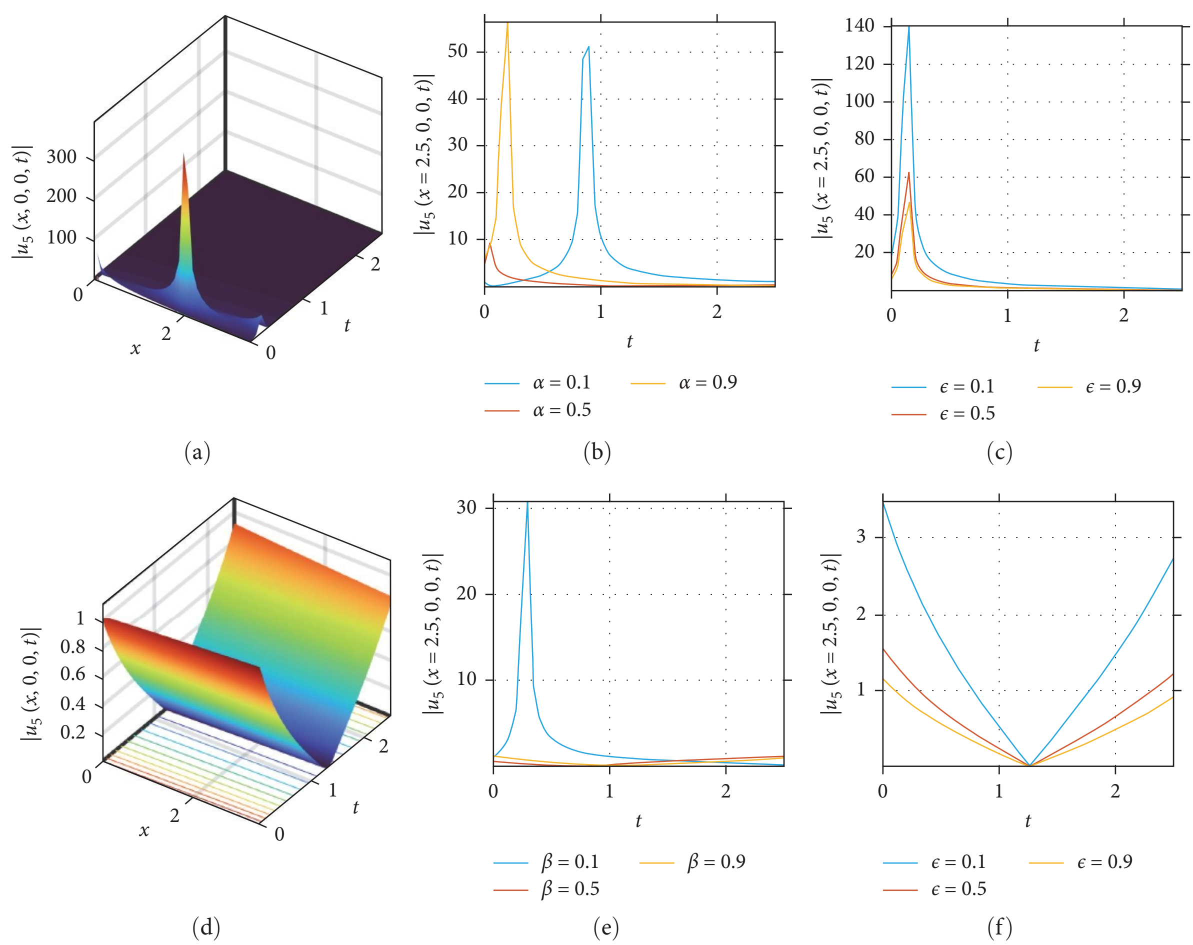

Figure 3: The solution $|u_5|$ for $\tau = 0.05, \psi = 1, \varepsilon = 1, \lambda = 1, \mu = 2, \nu = 1,$ (a) $\alpha = 0.8$, (b) $\alpha = 0.1, 0.5, 0.9$, (c) $\varepsilon = 0.1, 0.5, 0.9$, (d) $\beta = 0.89$, (e) $\beta = 0.1, 0.5, 0.9$, and (f) $\varepsilon = 0.1, 0.5, 0.9$.

0.5, 0.9. Figure 6c,f elucidates the variation in nature for $\varepsilon = 0.1$ and $\varepsilon = 0.1, 0.5, 0.9$ under two different conditions.

Figure 7a shows the solution for $(u_{13}(x, 0, 0, t))$ providing $\tau = 0.05\,\psi = 1, \varepsilon = 1, \lambda = 1, \mu = 2, \nu = 1$ and depict the shape for $\alpha = 0.7$ and Figure 7d for $\beta = 0.89$. Figure 7b,e shows how the shape reacts for the same values of α and β providing $\alpha, \beta = 0.1, 0.5, 0.9$ under two different frameworks considering other parameters unaffected. Variation in the nature of the soliton for $\varepsilon = 0.1$ in Figure 7c and for $\varepsilon = 0.1, 0.5, 0.9$ in Figure 7f is characterized.

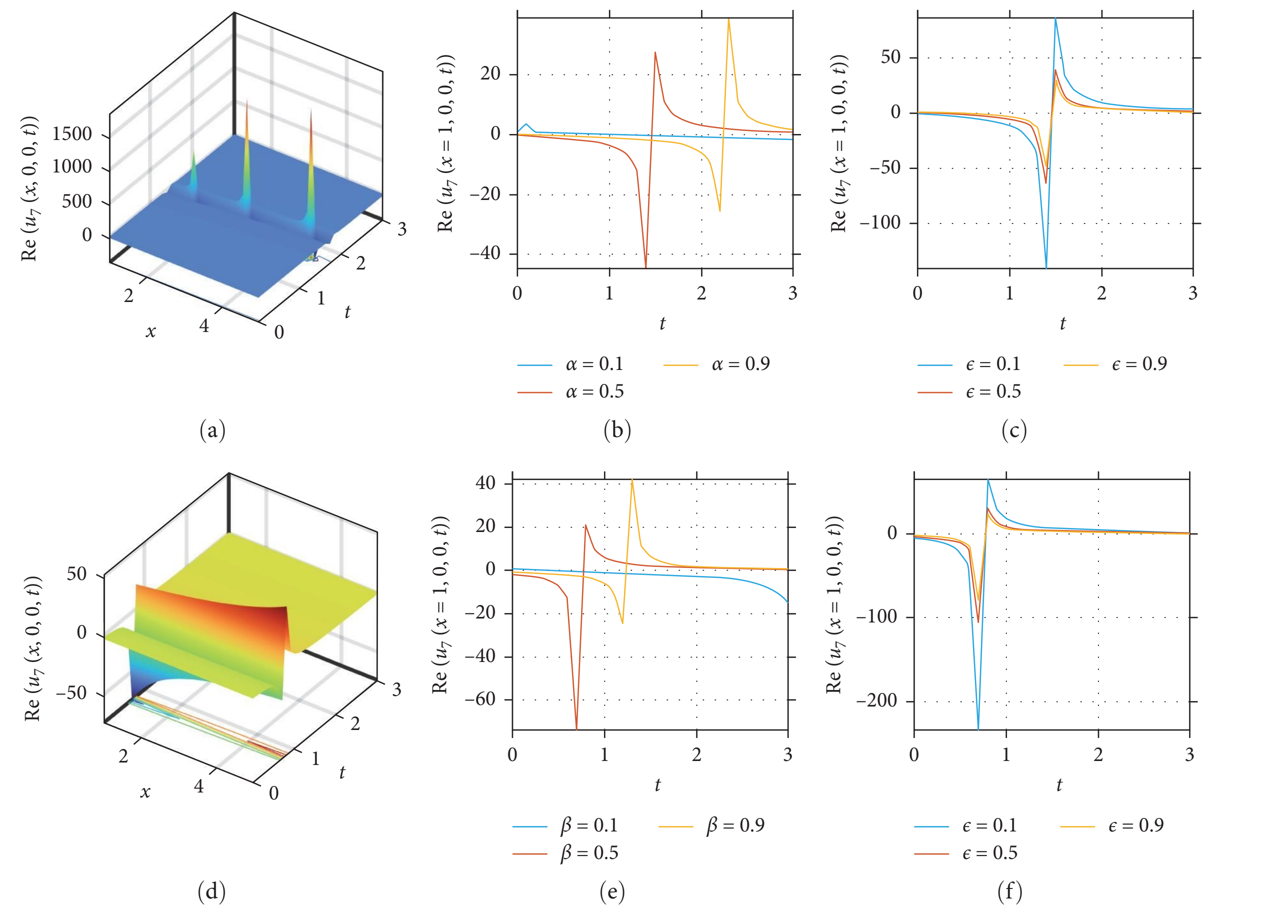

FIGURE 4: The solution Re(u_7) for $\tau = 0.05, \psi = 1, \varepsilon = 1, \lambda = 1, \mu = 2, \nu = 1$, (a) $\alpha = 0.5$, (b) $\alpha = 0.1, 0.5, 0.9$, (c) $\varepsilon = 0.1, 0.5, 0.9$, (d) $\beta = 0.89$, (e) $\beta = 0.1, 0.5, 0.9$, and (f) $\varepsilon = 0.1, 0.5, 0.9$.

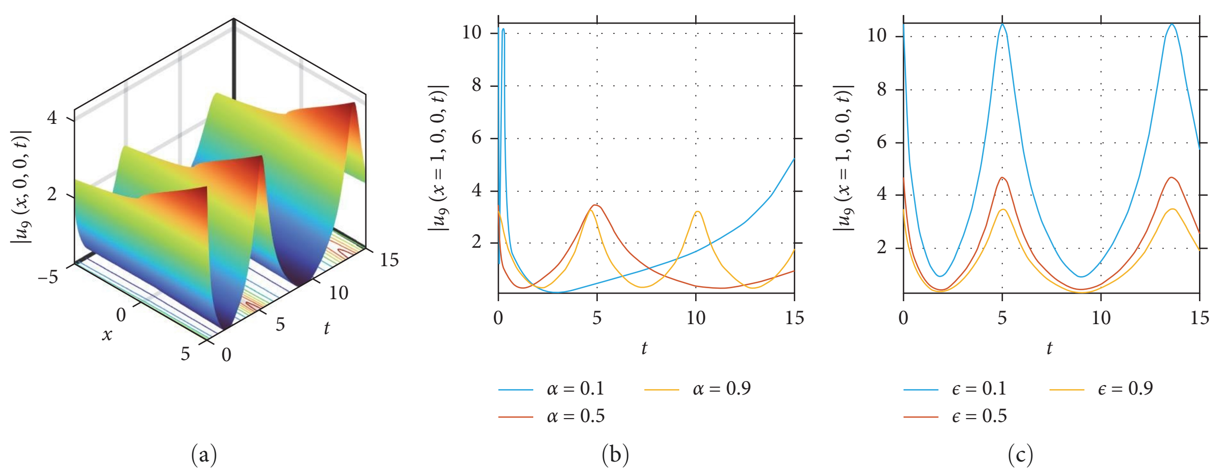

FIGURE 5: Continued.

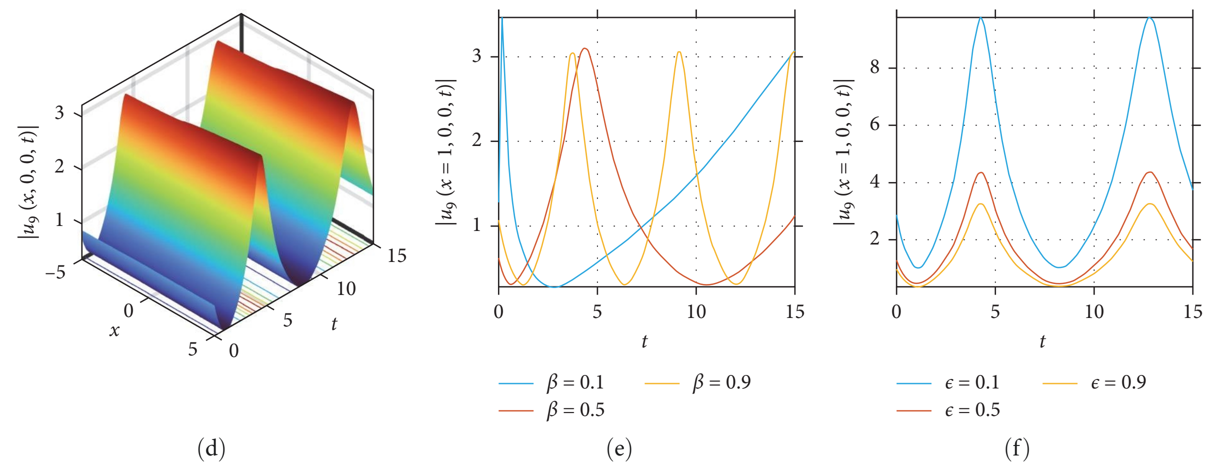

FIGURE 5: The solution $|u_9|$ for $\tau = 0.05, \psi = 1, \varepsilon = 1, \lambda = 3, \mu = 2, \nu = 1,$ (a) $\alpha = 0.7$, (b) $\alpha = 0.1, 0.5, 0.9$, (c) $\varepsilon = 0.1, 0.5, 0.9$, (d) $\beta = 0.89$, (e) $\beta = 0.1, 0.5, 0.9$, and (f) $\varepsilon = 0.1, 0.5, 0.9$.

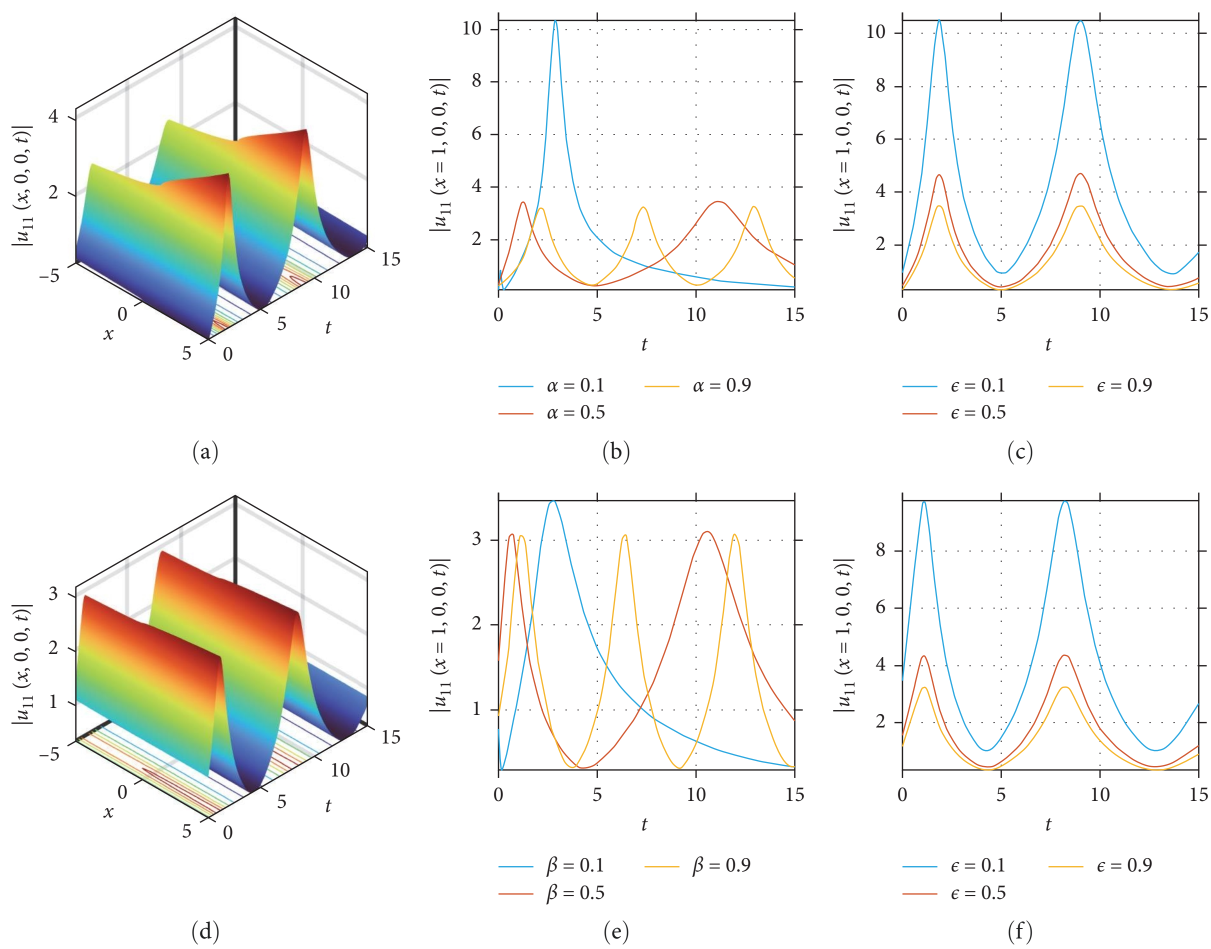

FIGURE 6: The solution $\mathrm{Re}(u_{11})$ for $\tau = 0.05, \psi = 1, \varepsilon = 1, \lambda = 3, \mu = 2, \nu = 1,$ (a) $\alpha = 0.7$, (b) $\alpha = 0.1, 0.5, 0.9$, (c) $\varepsilon = 0.1, 0.5, 0.9$, (d) $\beta = 0.89$, (e) $\beta = 0.1, 0.5, 0.9$, and (f) $\varepsilon = 0.1, 0.5, 0.9$.

In Figure 8, the soliton illustrates from the solution $(u_{17}(x, 0, 0, t))$ for the value of parameters $\tau = 0.05$, $\psi = 1$, $\varepsilon = 1$, $\lambda = 1$, $\mu = 0$, $\nu = 1$, $r = 1$. Figure 8a,d represents the breather wave which demonstrates how the soliton responds for different fractional derivatives providing $\alpha = 0.7$ for conformable fractional derivative and $\beta = 0.89$ for beta fractional derivative. Figure 8b,e represents the response of the graph for $\alpha, \beta = 0.1, 0.5, 0.9$, other parameters carry out the same values. Also, we show the effects of $\varepsilon = 0.1$ in terms of conformable derivative sense in Figure 8c and $\varepsilon = 0.1, \ 0.5, \ 0.9$ in terms of beta derivative sense in Figure 8f.

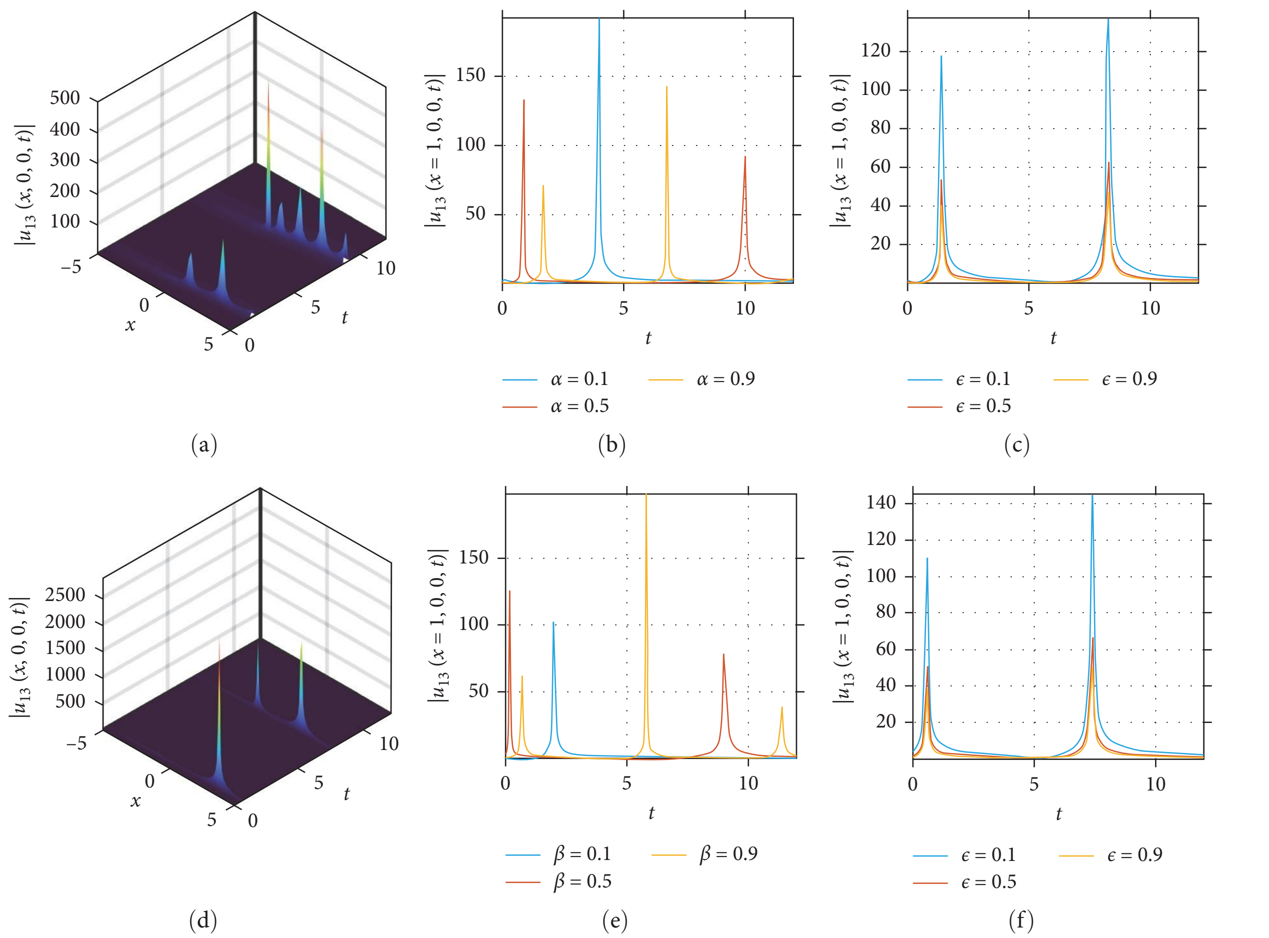

Figure 7: The solution $|u_{13}|$ for $\tau = 0.05, \psi = 1, \varepsilon = 1, \lambda = 1, \mu = 2, \nu = 1,$ (a) $\alpha = 0.7$, (b) $\alpha = 0.1, 0.5, 0.9$, (c) $\varepsilon = 0.1, 0.5, 0.9$, (d) $\beta = 0.89$, (e) $\beta = 0.1, 0.5, 0.9$, and (f) $\varepsilon = 0.1, 0.5, 0.9$.

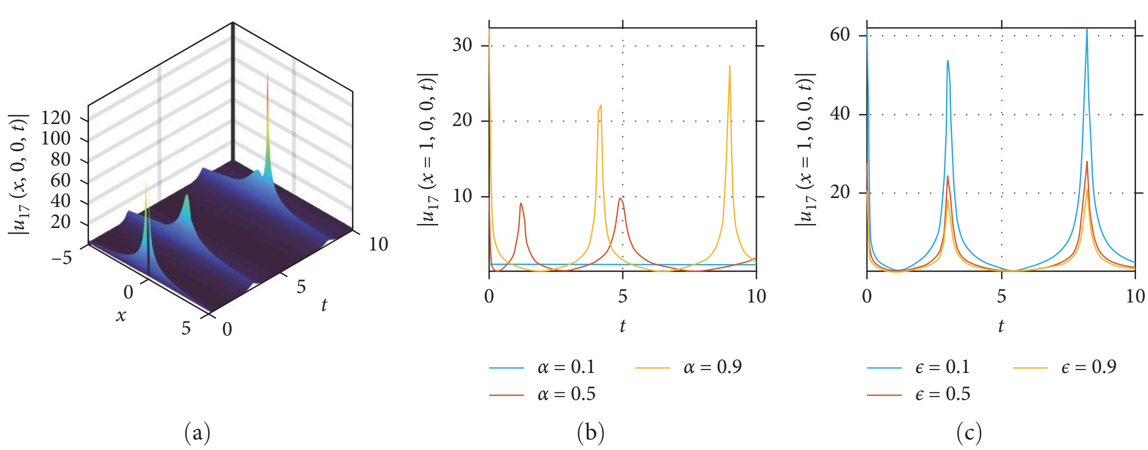

Figure 8: Continued.

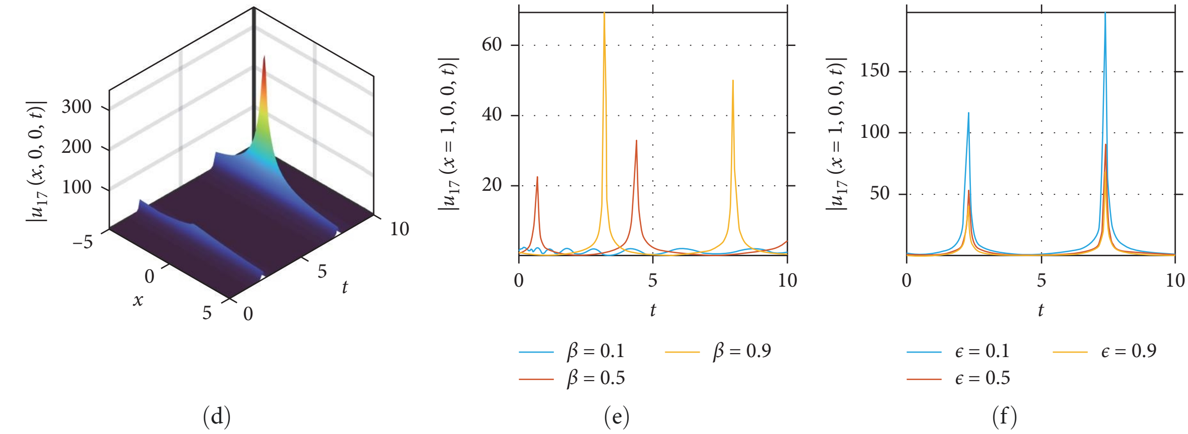

FIGURE 8: The solution $|u_{17}|$ for $\tau = 0.05, \psi = 1, \varepsilon = 1, \lambda = 1, \mu = 0, \nu = 1$, (a) $\alpha = 0.7$, (b) $\alpha = 0.1, 0.5, 0.9$, (c) $\varepsilon = 0.1, 0.5, 0.9$, (d) $\beta = 0.89$, (e) $\beta = 0.1, 0.5, 0.9$, and (f) $\varepsilon = 0.1, 0.5, 0.9$.

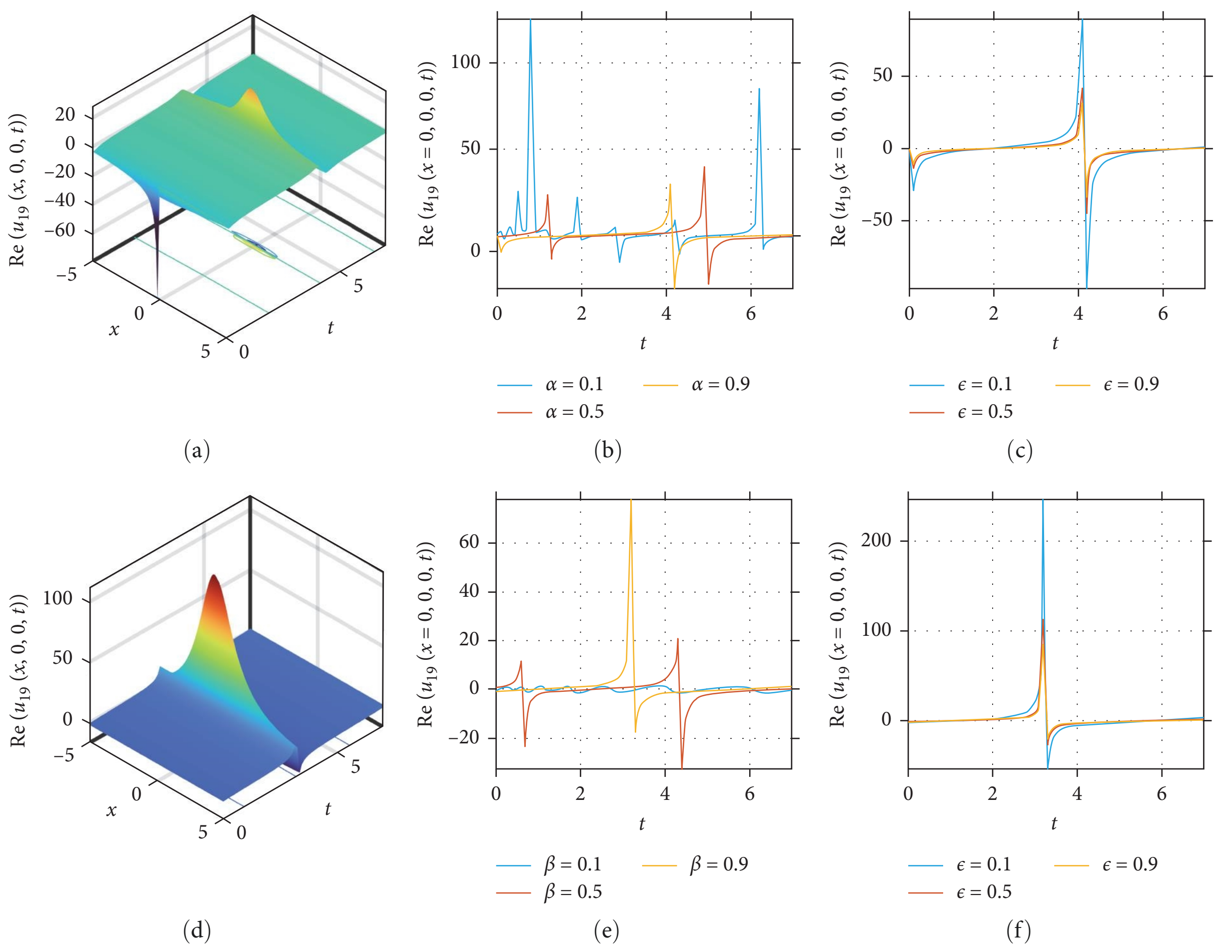

FIGURE 9: The solution $\mathrm{Re}(u_{19})$ for $\tau = 0.05, \psi = 1, \varepsilon = 1, \lambda = 1, \mu = 0, \nu = 1$, (a) $\alpha = 0.9$, (b) $\alpha = 0.1, 0.5, 0.9$, (c) $\varepsilon = 0.1, 0.5, 0.9$, (d) $\beta = 0.89$, (e) $\beta = 0.1, 0.5, 0.9$, and (f) $\varepsilon = 0.1, 0.5, 0.9$.

Figure 9a demonstrates the solution for $\mathrm{Re}(u_{19}(x, 0, 0, t))$ providing $\tau = 0.05$, $\psi = 1$, $\varepsilon = 1$, $\lambda = 1$, $\mu = 0$, $\nu = 1$, $r = 1$ and depict the shape for $\alpha = 0.9$ in conformable derivative sense. Figure 9b shows the variation in shape for different values of $\alpha = 0.1$, 0.5, 0.9 considering other parameters constant. Figure 9c depicts the change in shape for the effect of $\varepsilon = 0.1$. Figure 9d clarifies the graph for beta derivative sense providing $\beta = 0.89$ for same values of other parameters. Figure 9e highlights characterization in shape under beta derivative sense providing $\beta = 0.1$, 0.5, 0.9. Figure 9f illustrates the variation in shape for $\varepsilon = 0.1$, 0.5, 0.9 beneath beta derivative sense.

6. Comparisons

Various researchers have adopted different methods in different ways. Such as Islam and Akter [39] adopted the improved Tanh method along with the Riccati equation, and through conformable fractional derivatives, their outcomes were peakon, kink, singular kink, periodic, and anticompacton. Whereas the conformable fractional derivative was used with the (Nikiforov Unarov) NU method by Karayer et al. [40]. Kink, anti-kink, bright kink, singular periodic, and multiple singular periodic were findings of $(\frac{G'}{G'+G+A})$-expansion method with above mentioned derivative by Iqbal et al. [41]. Homotopy analysis transform method with Mittag–Leffler type kernel was used by Kumar and Baleanu [42] and they had surface graph. Whereas the surface graph was found by Liu et al. [43] through the homotopy perturbation method in the Caputo–Febrizio sense. Nagy [44] implemented the Sine method together with the shifted Chebyshev polynomials of the second kind in the Caputo sense. Caputo fractional derivative on extended cubic B spline was the exploration of Akram et al. [45]. Caputo sense on a q-homotopy analysis method. Using the q-homotopy analysis method in Caputo sense, Gao et al. [46] get kink and anti-kink. Through the Jumaries operator on (G'/G) – expansion method, Khan et al. [47] obtained periodic soliton. This type of soliton was also the result of Jumaries modified Riemann Liouville sense local fractional decomposition method by Merdan and Oral [48]. At the same time, modified Reimann–Liouville fractional derivative on solitary wave ansatz method adopted by Gunar and Bekir [49]. Rashid et al. [50] used $\exp(-(\varphi(\xi))$-expansion method with truncated M-fractional derivative and which provides multibell wave, multisoliton wave, and periodic rogue type wave. The above comparative analysis demonstrates that in any preceding exploration, the abstracted tactic has not been applied to the KG equation. Using the extended fractional Riccati expansion scheme, we derive a variety of dynamic optical wave patterns, including breather wave shapes, interactions between breather and king wave solitons, as well as bright and dark bell-shaped solutions. Additionally, we obtained different kinds of trigonometric and hyperbolic function solutions. These wave structures are illustrated through detailed pictorial representations. These testimonies are the novelty of the study. The above analysis with other existing research works has some unique, vigorous, and divergent solutions from previously conveyed consequences, which proves the effectiveness of the applied method.

7. Conclusions

In many previous works, different schemes associated with various fractional derivatives have been adopted to find traveling wave solutions of fractional KG equations. Through this research article, a new method has been applied to the fractional KG equation. Two types of fractional derivatives have been employed, namely, the conformable fractional derivative and the beta fractional derivative. The implementation provides some unique solutions to the abovementioned equation. The graphical representations for different values of α and β show the uniqueness of the solutions. The comparison chart demonstrates that the solutions are not available in previously reported works. Hence, the effectiveness of the employed method has been ascertained. We have used 2D plots, 3D plots, and contour plots to help explain the solutions that were found. The numerical results and comprehensive analysis, on the other hand, show that the proposed method is a sequential mathematical tool for studying nonlinear model singular, periodic, and soliton solutions. These plans work in concert, are efficient, and can be easily extended to solve other nonlinear models. We can reason that this strategy is so compelling and intense as to give improved arrangements. Using this approach, additional research projects can be completed.

Data Availability Statement

Data sharing is not applicable as no new data have been generated.

Conflicts of Interest

The authors declare no conflicts of interest.

Author Contributions

Md. Abde Mannaf: software, visualization, methodology, data curation, formal analysis. **Muktarebatul Jannah:** software, formal analysis, conceptualization, writing – original draft, editing. **Md. Habibul Bashar:** conceptualization, formal analysis, writing – review and editing, validation, project administration. **Md. Ekramul Islam:** resources, investigation, validation, project administration, supervision. **Zuel Rana:** software, formal analysis, review – original draft, editing. **Mst. Tania Khatun:** software, visualization, formal analysis. **Md. Noor-A-Alam Siddiki:** formal analysis, review – original draft. **Md. Shahinur Islam:** formal analysis, review – original draft, editing.

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