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Research article

Numerical and analytical solutions of new Blasius equation for turbulent flow

M. Mizanur Rahman [a], Shahansha Khan [b], M. Ali Akbar [c,*]

[a] Department of Computer Science and Engineering, Varendra University, Rajshahi, Bangladesh
[b] Department of Mathematics, Uttara University, Uttara Dhaka, Bangladesh
[c] Department of Applied Mathematics, University of Rajshahi, Bangladesh

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ABSTRACT

The Blasius equation for laminar flow comes from the Prandtl boundary layer equations. In this article, we establish a new and generic Blasius equation for turbulent flow derived from the turbulent boundary layer equation that can be used for turbulent as well as laminar flow. The analytical and numerical solutions have been investigated under specific conditions to the developed new Blasius equation. The analytical and numerical results have been compared through tables and graphs to validate the established model. In fluid dynamics, analytical solutions to complicated systems are tedious and time-consuming. Changing one or more constraints can introduce new challenges. In this case, symbolic computation software provides an easier and more flexible solution for fluid dynamical systems, even if boundary conditions are adjusted to explain reality. Therefore, the MATLAB code is used to investigate the new third-order Blasius equation. The comparison and graphical representations demonstrate that the achieved results are encouraging.

1. Introduction

Prandtl [1] derived the fundamental of boundary-layer theory in 1904, laying the groundwork for the unification of two hitherto disparate sciences: theoretical hydrodynamics and hydraulics. The fundamental application of boundary-layer theory is computing the skin-friction drag that acts on an object as it moves through a fluid, for example, the force exerted by an airplane wing, a turbine plate, or an entire ship [2]. Now, as the number of applications for microelectronics devices grows, boundary-layer theory has revived interest in the micro-scale analysis of gas and liquid flows, as demonstrated by Gadel-Hak [3] and Martin and Boyd [4]. Including these phenomena, in 1908, Blasius [5] proposed an equation to use the two dimensional basic governing equations of continuity, momentum of boundary layer for laminar flow, where the free stream velocity is constant. The boundary layer over a flat plate oriented parallel to the free flow provides the solution. The Blasius equation is a well-known third-order nonlinear equation that arises in fluid dynamics in definite boundary layer issues. The Blasius equation [6] for laminar flow has expanded to

$$a\,u'''(\eta) + u(\eta)\,u''(\eta) = 0, \tag{1.1}$$

with

* Corresponding author.
E-mail address: ali_math74@yahoo.com (M.A. Akbar).

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$$\left.\begin{matrix} u=0, u^{'}=0 \quad \text{when} \quad \eta=0 \\ u^{'}=1 \quad \text{when} \quad \eta=\infty \end{matrix}\right\},$$

where a is a constant, η is the dimensionless distance parameter, $u(\eta)$ is the dimensionless function of η and primes represent derivative with respect to η. It has an ordinary form and consists of the two types of classical Blasius equations for $a=1$ and $a=2$. In the previous articles, the shooting method was used to examine the nonlinear equation for $1 \leq a \leq 2$. Rahman [7] also extended the Blasius equation for laminar flow as follows:

$$u^{'''}(\eta)+\beta\, u(\eta)\, u''(\eta)=0, \tag{1.2}$$

with

$$\left.\begin{matrix} u=0, u^{'}=0 \quad \text{when} \quad \eta=0 \\ u^{'}=1 \quad \text{when} \quad \eta=\infty \end{matrix}\right\}, \tag{1.2a}$$

where β is the constant which is greater than zero because $\beta = c^2/2$, c is an arbitrary constant. He examined equation (1.2) along with (1.2a) analytically and numerically for a few unique circumstances which are laminar profile, parabolic profile, linear profile, sine-cosine profile and cubic profile, through using the finite difference method for numerical solutions and ingenious idea of Wang [8] for analytical solutions. The absence of the second derivative $u''(\eta)$ is the most significant stumbling block in solving the previously stated problem. Once this differential has been accurately predicted, the boundary value issue can be easily solved analytically. At the beginning of 1908, Eq. (1.2) was examined for $\beta=1$ and laminar flow by Blasius [5]. He derived the power series solution as follows:

$$u(\eta)=\sum_{i=0}^{\infty}(-\beta)^i \frac{B_i\, \sigma^{i+1}}{(3i+2)\,!}\eta^{3i+2}, \tag{1.3}$$

where $B_0 = B_1 = 1$, $B_i = \sum_{r=0}^{i-1}\binom{3i-1}{3r} B_r\, B_{i-r-1}$; $i \geq 2$, η is the dimensionless distance parameter, σ represent the unknown $u''(0)$ wherein (1.3) was computed only for a few terms. Howarth [9] investigated Eq. (1.2) for $\beta=0.5$ numerically and received $\sigma = 0.33206$. Asaithambi [10] studied the Blasius Eq. (1.2) for $\beta=0.5$ and obtained $\sigma=0.332057336$ more accurately and then investigated Eq. (1.2) by considering the condition $\beta=1$ and found $\sigma = 0.469600$. Wang [8] used an innovative concept to detect $u''(0)$ by analytically for $\beta = 1$. He used $x=u^{'}(\eta)$ and $z=u''(\eta)$ to transform Eq. (1.2) to another equation to examine straightforwardly, given below

$$z''+\beta\, x\,/\,z=0, x\,\varepsilon[0,1), \tag{1.4}$$

with $z(0) = u''(0)$, $z^{'}(0)=0$ and $\lim_{x\rightarrow} z(x) = 0$.

Wang applied the Adomian decomposition method to examine Eq. (1.4) for $\beta=1$ and got the following approximation

$$z(x)=\sigma-\frac{x^3}{6\sigma}-\frac{x^6}{180\sigma^3}-\frac{x^9}{2160\sigma^5}-\frac{x^{12}}{19008\sigma^7}\cdots, \tag{1.5}$$

Wang [8] examined this equation up to six terms of the series Eq. (1.5) and got this result $u''(0) = \sigma = 0.453539$. The Adomian decomposition method (ADM) was used by Hashim [11] and increased the terms of series (1.5) up to x^{24}. He then approximated this function using the {12/12} diagonal Pade approximation. The Pade approximation was utilized by Faiz and Wafaa [12] up to {23/23}. Pade approximation and got the outcome $u''(0) = 0.469009$. All of the above authors examined the Blasius equation for laminar flow.

The Adomian decomposition technique has been exploited to find the approximate results of a large class of differential, integral, and integro-differential equations [13–17]. The technique gives the result in a swiftly convergent series with materials which are elegantly calculated. Abdelali and Rachid [18] studied the Falkner and Skan equation:

$$u^{'''}+u\,u''+\beta\left(1-u^{'2}\right)=0, \tag{1.6}$$

with

$$\left.\begin{matrix} u=0, u^{'}=0 \quad \text{when} \quad \eta=0 \\ u^{'}=1 \quad \text{when} \quad \eta=\infty \end{matrix}\right\} \tag{1.6a}$$

as the generalized Blasius equation which is also generated for laminar flow. They have investigated the existence and uniqueness of the solution of Eq. (1.6) with initial condition (1.6a) to apply nonstandard analysis techniques. Some Author investigated Eq. (1.6) for several values of β. For example, when $0 < \beta \leq 1$, Weyl [19] proves the existence a classical solution and its uniqueness and when $1 < \beta$, Craven and Peletier [20] investigated the existence of an overshoot solution by numerically. And what more, many authors examined the Blasius equation by using several numerical methods, such as perturbation method [21–23], Adomian decomposition [24,25] method, Variational iteration technique [26–32], etc.

Ramzan [33] investigated the temperature-dependent fluctuation features of viscosity and thermal conductivity in nanofluid flow over a rotating disk using a modified Fourier law and graphically portrayed the ascending pertinent parameter by using Bvp4c (Boundary Value Problems), a built-in MATLAB function. Kouz [34] applied the finite element method to simulate the heat transfer and irreversible in a two-phase mixed convection current through a corrugated envelope filled with an aluminum-alumina liquid and contains a rotating solid cylinder in the presence of a uniform magnetic field. Sohail [35] used the standard turbulence model κ-ε to obtain an accurate analysis of the controlled turbulent combustion, which is mainly fitting for systems of solar power in the oil industry. Shirvan [36] employed the Darcy-Brinkman-Forchheimer and the k-ε turbulent models to achieve the heat transfer and the heat exchanger efficiency of the considered model. Ali et al. [37] used the basic governing equations involving momentum, continuity, heat, and induced magnetic field to investigate the magneto-hydrodynamic laminar flow in which mixture nano-particles with heat transfer phenomenon over a stretching sheet absorbed in a porous medium and the effect of the brought magnetic field was also taken into account. Rasheed [38] used the basic thermal boundary layer equation and mass for the unsteady flow to examine three-dimensional thin film flow over an angular rotating disk plane in the presence of nano-particles. They have established the differential equations from the governing equations and then obtained the series and numerical solutions with the help of the homotopy asymptotic method (HAM) and the BVPh2-midpoint method respectively. Akbarzadeh [39] investigated the impact of the simultaneous application of corrugated walls and nano-particles on the performance of solar heaters using the elementary governing equations of momentum, continuity, energy, and volume fraction. Jha and Gombo [40,41] investigated the influence of an exponentially decaying or growing time-dependent pressure gradient on unsteady Dean flow in a curved concentric cylinder and developed a laminar Dean flow with an oscillating time-dependent pressure gradient using the Navier-Stokes equation. The Navier-Stokes equations are a system of equations that contain the momentum equation and continuity equation. Idowu and Falodun [42] studied the thermophoresis effect on heat and mass transfer flow of magneto-hydrodynamics non-Newtonian nanofluid.

In the case of turbulent flow, the Blasius equation has a significant role, yet it was not designed for turbulent flow to our optimal understanding. Therefore, the aim of this article is to develop a new and general Blasius equation using the two-dimensional basic governing equations of momentum and continuity of boundary layer for turbulent flow that can be used for turbulent flow as well as for laminar flow. Under specific conditions, the developed new Blasius equation has been investigated analytically and numerically. The analytical and numerical results have been compared through tables and graphs to validate the established model.

2. The Generalized blasius Equation for Turbulent flow

Turbulence is one of the utmost challenging issues in the natural sciences and cannot be fully described by current physical and mathematical theory. Because, the turbulent flow is characterized the chaotic behavior (detailed can be found in Balonishnikov [43], Dombre et al. [44], thus this phenomena is not discussed in detailed here), instantaneous velocities, and pressure fields fluctuation considerably. Therefore, it is important to probe flow statistics. One of the most widely used techniques when dealing with turbulence is the application of the Reynolds decomposition. The features of the transitory flow are divided into the mean and the variable parts. It is assumed that the mean value of the variable part is always zero. Thus, the boundary layer equation produces a fully turbulent boundary layer equation. The turbulent boundary layer equations (Skote et al. [45]) with constant pressure, steady (time independent flow) and an incompressible fluid are:

$$\frac{\partial u_m}{\partial x}+\frac{\partial v_m}{\partial y}=0, \tag{2.1}$$

$$u_m\frac{\partial u_m}{\partial x}+v_m\frac{\partial u_m}{\partial y}=\frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u_m}{\partial y}-\langle\rho u_m^{'}v_m^{'}\rangle\right), \tag{2.1a}$$

where x, y are the lengths of the fluid along and perpendicular to the wall, u_m is the mean stream-wise velocity, v_m is the mean wall-normal velocity, ρ is the density, μ is the dynamic viscosity, U_t is the free-stream velocity and $<\rho u_m^{'}v_m^{'}>$ is the Reynolds shear stress or eddy shear stress of the fluid. When the fluid at rest that means near to the wall then mean stream-wise and mean wall-normal velocities are both zero. Again the fluid moves away from the wall or out of boundary layer that means y tends to infinite then the mean stream-wise velocity goes to free-stream velocity. Finally, we obtain the boundary condition of the boundary layer Eq. (2.1a) as follows:

$$\left.\begin{array}{ll} \mathrm{u_m}=v_m=0 & \text{when } \mathrm{y}=0 \\ \mathrm{u_m}\rightarrow \mathrm{U_t} & \text{when } \mathrm{y}\rightarrow\infty \end{array}\right\}, \tag{2.1b}$$

Now we discuss how to make the new Blasius equation from Eq. (2.1a) with the help to Eq. (2.1b). The resulting turbulent boundary layer equations cannot be solved without a closure assumption which relates the turbulent shear stress to the mean flow variables. At first, we will establish a new Blasius equation applicable to turbulent flow with the help of Rahman [7] discussed in the underneath. Boussinesq (Hassan [46]) proposed that $\langle\rho\, u_m^{'}\, v_m^{'}\rangle=\mu_t\,\frac{\partial u_m}{\partial y}$, where μ_t is the turbulent shear viscosity or eddy viscosity. There is a key difference between μ and μ_t such that $\mu\ll\mu_t$, μ is a property of the fluid and is a function of the temperature, while μ_t is a function of the flow and its value depending on the initial and boundary conditions of the problem under consideration. According to Boussinesq's concept, the eddy viscosity μ_t is a scalar value. Thus, Eqs. (2.1) and (2).1a) become

$$\frac{\partial u_m}{\partial x}+\frac{\partial v_m}{\partial y}=0, \tag{2.2}$$

$$u_m \frac{\partial u_m}{\partial x} + v_m \frac{\partial u_m}{\partial y} = \frac{(\mu + \mu_t)}{\rho} \frac{\partial^2 u_m}{\partial y^2}, \tag{2.2a}$$

with boundary conditions

$$\left.\begin{array}{l} \mathrm{u_m} = v_m = 0 \quad \text{when } \mathrm{y} = 0 \\ \mathrm{u_m} \rightarrow \mathrm{U_t} \quad \text{when } \mathrm{y} \rightarrow \infty \end{array}\right\}. \tag{2.2b}$$

Now, we transform Eq. (2.2) and Eq. (2.2a) to suitable equations by using the stream function ψ

$$u_m = -\frac{\partial \psi}{\partial y}, v_m = \frac{\partial \psi}{\partial x}. \tag{2.3}$$

Then, Eq. (2.3) spontaneously satisfies Eq. (2.2).

The order of the boundary layer thickness is $\left(\frac{\mu_t x^{n-1}}{\rho U_t}\right)^{1/n}$; n > 2 approximate for turbulent flow, i.e.

$$\delta \approx \left(\frac{\mu_t x^{n-1}}{\rho U_t}\right)^{1/n}, \text{then } \delta = c\left(\frac{\mu_t x^{n-1}}{\rho U_t}\right)^{1/n}, \tag{2.4}$$

where c is an arbitrary constant. The unfamiliar c in Eq. (2.4) can be determined. Hence, let us consider a new dimensionless distance parameter to be $\eta = y/\delta$,

$$\text{wherein } \eta = \frac{y}{c\left(\frac{\mu_t x^{n-1}}{\rho U_t}\right)^{1/n}}, \tag{2.5}$$

in order to the similarity method, let

$$\frac{u_m}{U_t} = F(\eta), \tag{2.6}$$

be the velocity profile.

By means of Eqs. (2.5) and (2.6) into Eq. (2.3), the stream function is as follows:

$$\psi = -\int u_m dy = -c\left(\frac{\mu_t}{\rho}\right)^{\frac{1}{n}} (U_t x)^{\frac{n-1}{n}} u(\eta), \tag{2.7}$$

where $u(\eta) = \int F(\eta)\, d\eta$. Then Eq. (2.6) becomes

$$u_m = U_t u'(\eta). \tag{2.8}$$

Similarly, using Eq. (2.5) and Eq. (2.7) into Eq. (2.3) and after simplification, we found

$$v_m = \frac{c(n-1)}{n} \frac{U_t^{\frac{n-1}{n}} \left(\frac{\mu_t}{\rho}\right)^{\frac{1}{n}}}{x^{\frac{1}{n}}} \left[\eta u'(\eta) - u(\eta)\right], \tag{2.9}$$

$$\frac{\partial u_m}{\partial x} = -\frac{(n-1)}{n} \frac{U_t}{x} \eta\, u''(\eta), \tag{2.10}$$

$$\frac{\partial u_m}{\partial y} = \frac{1}{c} \frac{U_t^{\frac{n+1}{n}}}{\left(\frac{\mu_t}{\rho}\right)^{\frac{1}{n}}} x^{\frac{1-n}{n}} u''(\eta), \tag{2.11}$$

and

$$\frac{\partial^2 u}{\partial y^2} = = \frac{1}{c^2} \frac{U_t^{\frac{n+2}{n}}}{\left(\frac{\mu_t}{\rho}\right)^{\frac{2}{n}}} x^{\frac{2(1-n)}{n}} f'''(\eta). \tag{2.12}$$

Substituting Eqs. (2.8)-(2.12) into Eq. (2.2a), after simplification; we found the following equation

$$u'''(\eta) + \frac{\beta \mathrm{Re}_t^{\frac{n-2}{n}}}{\left(1 + \frac{\mu}{\mu_t}\right)} u(\eta)\, u''(\eta) = 0, \tag{2.13}$$

where $\beta = \frac{c^2(n-1)}{n}$ and $\mathrm{Re}_t = \frac{\rho U_t x}{\mu_t}$ (Reynolds number for turbulent flow) which is also called parameter of Eq. (2.13). If the eddy viscosity is much higher than the fluid viscosity, *i. e.* $\mu \ll \mu_t$ or $\frac{\mu}{\mu_t} \ll 1$, the term $\frac{\mu}{\mu_t}$ should be negligible. This implies that Eq. (2.13) becomes

$$u^{'''}(\eta)+\beta \mathrm{Re}_t^{\frac{n-2}{n}}u(\eta)\,u''(\eta)=0, \tag{2.14}$$

which is the new general Blasius equation for turbulent flow, depends on only Reynolds number. If $n=2$, then the reduction of Eq. (2.14) to the general Blasius equation for laminar flow which is shown in Rahman [7].

Since c is not equal to zero, n is natural number, $n>2$ and $\beta=\frac{c^2(n-1)}{n}$, then $\beta>0$. Using Eq. (2.2b) in Eq. (2.5), it is observed that $y=0$ implies $\eta=0$ and $y\to\infty$ implies $\eta\to\infty$. Then from Eqs. (2.8) and (2.9), we obtain $u_m=0$ and $v_m=0$ for $y=0$, implies $u=0$ and $u^{'}=0$ for $\eta=0$. Again, using Eq. (2.2b) in Eq. (2.8), we get $u_m\to U_t$, this implies that $u^{'}=1$. Hence the boundary conditions can be re-written as follows

$$\left.\begin{array}{l} u=0, u^{'}=0 \quad \text{when} \quad \eta=0 \\ u^{'}=1 \quad \text{when} \ \ \eta\to\infty \end{array}\right\}.$$

Therefore, finally, we establish

$$u^{'''}+\beta \mathrm{Re}_t^{\frac{n-2}{n}}u\,u''=0, \tag{2.15}$$

where $\beta>0$ and boundary conditions are

$$\left.\begin{array}{l} u=0, u^{'}=0 \quad \text{when} \quad \eta=0 \\ u^{'}=1 \quad \text{when} \ \ \eta\to\infty \end{array}\right\}. \tag{2.15a}$$

The general Blasius equation for turbulent flow is Eq. (2.15), together with (2.15a).

3. The series solution: The general blasius equation

The new general Blasius equation is as follows

$$u^{'''}+\beta\,\mathrm{Re}_t^{\frac{n-2}{n}}u\,u''=0, \tag{3.1}$$

with

$$\left.\begin{array}{l} u=0, u^{'}=0 \quad \text{when} \quad \eta=0 \\ u^{'}=1 \quad \text{when} \ \ \eta\to\infty \end{array}\right\}. \tag{3.1a}$$

Since Eq. (3.1) is a third-order nonlinear equation with three boundary conditions provided in Eq. (3.1a), it is sufficient to establish the result completely, but a closed-form solution of Eq. (3.1) is unattainable. Thus, the series solution can be found in the same way that Blasius [5] did, as shown below:

$$u(\eta)=\frac{\sigma}{2\;!}\eta^2-\frac{k\sigma^2}{5\;!}\eta^5+\frac{11k^2\sigma^3}{8\;!}\eta^8-\frac{375k^3\sigma^4}{11\;!}\eta^{11}+\ldots \tag{3.2}$$

where $k=\beta \mathrm{Re}_t^{\frac{n-2}{n}}$.

Therefore, series (3.2) can be written as

$$u(\eta)=\sum_{i=0}^{\infty}(-k)^i\frac{B_i\,\sigma^{i+1}}{(3i+2)\;!}\eta^{3i+2}, \tag{3.3}$$

where $B_0=B_1=1$, $B_i=\sum_{r=0}^{i-1}\binom{3i-1}{3r}B_rB_{i-r-1}$; $i\geq 2$ and σ signifies the indefinite $u''(0)$ which is analytic (series) solution of the new general Blasius equation for turbulent flow.

Recently, Wang [8] used an inventive idea to find out $u''(0)=\sigma$ analytically for $\beta=1$ and also used $x=u^{'}(\eta)$ and $z=u''(\eta)$ to transmute Eq. (3.1) to alternative equation to examine easily as follows:

$$z''+k\,(x\,/\,z)=0, \tag{3.4}$$

with initial conditions

$$z(0)=u''(0), z^{'}(0)=0 \text{ and } \lim_{x\to} z(x)=0. \tag{3.5}$$

It should be noted that Eq. (3.4) is a nonlinear equation, while the three boundary constraints presented in Eq. (3.5) are adequate to entirely specify the solution. The general outcome of Eq. (3.4) is unattainable in closed form. Thus, to solve (3.4) with the boundary constraints (3.5) for β and Re_t, the series solution of (3.4) is to be assumed of the form:

$$z(x)=\sum_{i=0}^{\infty}a_i\,x^i, a_0\neq 0. \tag{3.6}$$

Inserting (3.6) into Eq. (3.4), we obtain

$$\sum_{i=2}^{\infty} i(i-1)\, a_i x^{i-2} \times \left(\sum_{i=0}^{\infty} a_i x^i\right) + \beta\, \mathrm{Re}_t^{\frac{n-2}{n}}\, x = 0, \tag{3.7}$$

is an identity equation. Thus, using MATLAB from (3.7), we obtain the subsequent relations:

$$a_1 = a_2 = a_4 = a_5 = a_7 = a_8 = \dots = 0,$$

where $a_3 = -\frac{k}{6\sigma}, a_6 = -\frac{k^2}{1.80E2\sigma^3}, a_9 = -\frac{k^3}{2.16E3\sigma^5}, a_{12} = -\frac{k^4}{1.90E4\sigma^7}, a_{15} = -\frac{2.09E3k^5}{2.99E8\sigma^9},$.

$$a_{18} = -\frac{3.14E4k^6}{3.05E10\sigma^{11}}, a_{21} = -\frac{4.61E4k^7}{2.85E11\sigma^{13}}, a_{24} = -\frac{6.27E7k^8}{2.36E15\sigma^{15}}, a_{27} = -\frac{2.12E9k^9}{4.67E17\sigma^{17}},$$

$$a_{30} = -\frac{1.62E12k^{10}}{2.03E21\sigma^{19}}, a_{33} = -\frac{1.50E11k^{11}}{1.05E21\sigma^{21}}, a_{36} = -\frac{2.94E15k^{12}}{1.13E26\sigma^{23}}, a_{39} = -\frac{4.58E18k^{13}}{9.46E29\sigma^{25}},$$

$$a_{42} = -\frac{1.78E20k^{14}}{1.96E32\sigma^{27}}, a_{45} = -\frac{1.73E21k^{15}}{9.98E33\sigma^{29}}, a_{48} = -\frac{1.49E25k^{16}}{4.49E38\sigma^{31}}, a_{51} = -\frac{4.58E30k^{17}}{7.12E44\sigma^{33}},$$

$$a_{54} = -\frac{2.58E31k^{18}}{2.06E46\sigma^{35}}, a_{57} = -\frac{1.78E34k^{19}}{7.22E49\sigma^{37}}, \text{and } a_{60} = -\frac{8.79E37k^{20}}{1.80E54\sigma^{39}}.$$

Then, substituting these values into solution (3.6), we found

$$\begin{aligned} z(x) = \sigma - \frac{kx^3}{6\sigma} - \frac{k^2x^6}{1.80E2\sigma^3} - \frac{k^3x^9}{2.16E3\sigma^5} - \frac{k^4x^{12}}{1.90E4\sigma^7} - \frac{2.10E3k^5x^{15}}{2.99E8\sigma^9} - \frac{3.14E4k^6x^{18}}{3.05E10\sigma^{11}} \\ - \frac{4.61E4k^7x^{21}}{2.85E11\sigma^{13}} - \frac{6.27E7k^8x^{24}}{2.36E15\sigma^{15}} - \frac{2.12E9k^9x^{27}}{4.67E17\sigma^{17}} - \frac{1.62E12k^{10}x^{30}}{2.03E21\sigma^{19}} - \frac{1.50E11k^{11}x^{33}}{1.05E21\sigma^{21}} \\ - \frac{2.94E15k^{12}x^{36}}{1.13E26\sigma^{23}} - \frac{4.58E18k^{13}x^{39}}{9.46E29\sigma^{25}} - \frac{1.78E20k^{14}x^{42}}{1.96E32\sigma^{27}} - \frac{1.73E21k^{15}x^{45}}{9.98E33\sigma^{29}} - \frac{1.49E25k^{16}x^{48}}{4.49E38\sigma^{31}} \\ - \frac{4.58E30k^{17}x^{51}}{7.12E44\sigma^{33}} - \frac{2.58E31k^{18}x^{54}}{2.06E46\sigma^{35}} - \frac{1.78E34k^{19}x^{57}}{7.22E49\sigma^{37}} - \frac{8.79E37k^{20}}{1.80E54\sigma^{39}} \dots, \end{aligned} \tag{3.8}$$

where σ denotes the unknown $u''(0)$, This is the series solution to the new generic Blasius equation. We can estimate by truncating the series solution (3.8) after twenty-one terms for $x = 1$, putting (3.5) into Eq. (3.8), and solving the equation with MATLAB, we can derive an estimated value of σ.

$$\begin{aligned} \sigma^{40} - \frac{k\sigma^{38}}{6} - \frac{k^2\sigma^{36}}{1.80E2} - \frac{k^3\sigma^{34}}{2.16E3} - \frac{k^4\sigma^{32}}{1.90E4} - \frac{2.10E3k^5\sigma^{30}}{2.99E8} - \frac{3.14E4k^6\sigma^{28}}{3.05E10} \\ - \frac{4.61E4k^7\sigma^{26}}{2.85E11} - \frac{6.27E7k^8\sigma^{24}}{2.36E15} - \frac{2.12E9k^9\sigma^{22}}{4.67E17} - \frac{1.62E12k^{10}\sigma^{20}}{2.03E21} - \frac{1.50E11k^{11}\sigma^{18}}{1.05E21} \\ - \frac{2.94E15k^{12}\sigma^{16}}{1.13E26} - \frac{4.58E18k^{13}\sigma^{14}}{9.46E29} - \frac{1.78E20k^{14}\sigma^{12}}{1.96E32} - \frac{1.73E21k^{15}\sigma^{10}}{9.98E33} - \frac{1.49E25k^{16}\sigma^8}{4.49E38} \\ - \frac{4.58E30k^{17}\sigma^6}{7.12E44} - \frac{2.58E31k^{18}\sigma^4}{2.06E46} - \frac{1.78E34k^{19}\sigma^2}{7.22E49} - \frac{8.79E37k^{20}}{1.80E54} = 0\,. \end{aligned} \tag{3.9}$$

By using MATLAB, from (3.9), we obtain value of σ which is 1.9842 for $\beta = 0.021942857$, $n = 7$ and $\mathrm{Re}_t = 10000$.

4. The numerical solution: The new general blasius equation

The generic Blasius equation is a nonlinear differential equation and thus it is generally difficult to attain analytical solutions. Therefore, we will investigate the numerical solutions by means of the finite-difference method using computer software which provides a simple and flexible solution even if the boundary conditions are modified to explain the reality. The finite-difference methods (FDM) are a group of numerical techniques for solving differential equations by approximating derivatives with finite differences in numerical analysis. So, we have used the finite difference methods and then MATLAB code to solve the new generic Blasius equation. The new generic Blasius equation takes the subsequent shape after using Wang's concept [8]:

$$z'' + k\,(x/z) = 0, \text{where } k = \beta Re_t^{\frac{n-2}{n}}, \tag{4.1}$$

with initial conditions

$$z(0) = u''(0), z'(0) = 0 \text{ and } \lim_{x \to 1} z(x) = 0. \tag{4.1a}$$

We use the finite difference method as follows:

Table 1
Numerical and analytical results for the values of Re_t, $\beta = 0.021942857$ and $n = 7$.

Reynolds no. Re_t	The value of k or $\beta Re_t^{\frac{n-2}{n}}$ for $n = 7$ $\beta = 0.021942857$	Series solution of $u''(0)$	Numerical solution of $u''(0)$	Errors of Percentage (%)
5000	9.6253	1.5490	1.5399	0.59
10,000	15.792	1.9842	1.9724	0.60
20,000	25.9094	2.5415	2.5264	0.60
30,000	34.6128	2.9375	2.9201	0.60
40,000	42.5088	3.2553	3.2361	0.59
50,000	49.854	3.5254	3.5045	0.60
60,000	56.7882	3.7626	3.7403	0.60
70,000	63.3983	3.9755	3.9520	0.59
80,000	69.7429	4.1697	4.1450	0.60
90,000	75.8644	4.3489	4.3231	0.60
100,000	81.7941	4.5156	4.4889	0.59
110,000	87.5564	4.6720	4.6443	0.60
120,000	93.1708	4.8195	4.7909	0.60
130,000	98.6529	4.9592	4.9298	0.60
140,000	104.0157	5.0922	5.0621	0.59
150,000	109.2701	5.2193	5.1883	0.60
160,000	114.4252	5.3410	5.3093	0.60
170,000	119.4891	5.4579	5.4255	0.60
180,000	124.4684	5.5704	5.5374	0.60
190,000	129.3694	5.6790	5.6454	0.60
200,000	134.1971	5.7840	5.7498	0.59

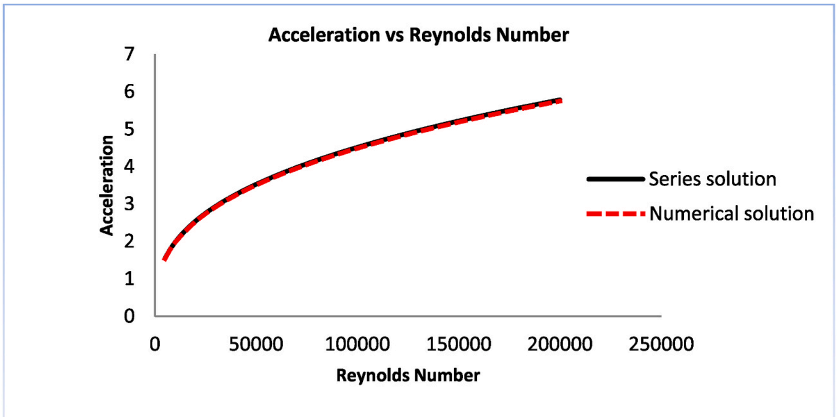

Fig. 1. Acceleration vs Reynolds number to increase acceleration how effect Reynolds number for $\beta = 0.021942857$ and $n = 7$.

$$z' = \frac{z_{j+1} - z_{j-1}}{2h} \text{ and } z'' = \frac{z_{j+1} - 2z_j + z_{j-1}}{h^2}, \tag{4.2}$$

where $h = x_j - x_{j-1}$, and j is the positive integer.

By means of (4.2), from Eq. (4.1) together the conditions provided in (4.1a), we establish

$$\frac{z_{j+1} - 2z_j + z_{j-1}}{h^2} + k\frac{x_j}{z_j} = 0$$

Therefore,

$$z_j \left(z_{j+1} - 2z_j + z_{j-1}\right) + k\, x_j h^2 = 0. \tag{4.3}$$

We make six subinterval from the interval [0, 1) such that $h = 0.78$ and $x_0 = 0$, $x_j = x_{j-1} + h$, $j = 0,1,2,3,4,5$; $x_6 = 1$ and $z_0 = \sigma$, $z_6 = 0$.

We have considered the values of j from 0 to 5 in Eq. (4.3), thus we obtain six equations. We have made a function in MATLAB and we got the result after calling this function in MATLAB which the value of σ for $\beta = 0.021942857$, $n = 7$ and $Re_t = 10000$ is equal to 1.9724 which is approximately equal to the above series solution.

In the same way, we found the results of $u''(0)$ for several Reynolds numbers, $\beta = 0.021942857$, $n = 7$ and also $\beta = 0.1125$, $n = 5$ by series solution and numerical solution given in Table 1:

Table 2
Numerical and analytical results for the values of Re_t, $\beta = 0.11552$ and $n = 5$.

Reynolds no. Re_t	The value of k or $\beta\,\mathrm{Re}_t^{\frac{n-2}{n}}$ for $n = 5$ $\beta = 0.11552$	Series solution of $u''(0)$	Numerical solution of $u''(0)$	Errors of Percentage (%)
5000	19.1443	2.1846	2.1717	0.59
10,000	29.0173	2.6896	2.6737	0.59
20,000	43.982	3.3113	3.2917	0.6
30,000	56.0957	3.7396	3.7174	0.6
40,000	66.6643	4.0767	4.0525	0.6
50,000	76.2148	4.3589	4.3331	0.6
60,000	85.0253	4.604	4.5767	0.6
70,000	93.2644	4.8219	4.7933	0.6
80,000	101.0441	5.019	4.9892	0.6
90,000	108.4433	5.1995	5.1687	0.6
100,000	115.52	5.3664	5.3347	0.59
110,000	122.3187	5.5221	5.4894	0.6
120,000	128.8742	5.6681	5.6346	0.59
130,000	135.2145	5.8059	5.7715	0.6
140,000	141.3624	5.9364	5.9013	0.59
150,000	147.337	6.0606	6.0247	0.6
160,000	153.1543	6.1791	6.1425	0.6
170,000	158.8278	6.2925	6.2552	0.6
180,000	164.3693	6.4013	6.3634	0.6
190,000	169.7889	6.506	6.4674	0.6
200,000	175.0956	6.6069	6.5677	0.6

5. Results and discussion

Cortell [6] and Rahman [7] extended the Blasius equation to Eq. (1.1) and Eq. (1.2). In this article, we have developed a new Blasius equation (2.15) which depends on the Reynolds number. It is known that the flows at Reynolds numbers greater than 4000 are usually turbulent, while Reynolds number below 2300 generally remains laminar. Flow in the range of Reynolds numbers 2300 to 4000 and recognized as transition. Then, the range of Re_t greater than four thousand to infinite, *i.e.* $4000 < Re_t < \infty$ is for turbulent flow. It is observed from Figs. 1 and 2 that the acceleration increased when the Reynolds number increased and also it is observed that the acceleration has less increased for increment of integer number n which is shown at Fig. 3.

From Tables 1 and 2, it is seen that the analytical and numerical results are almost same and the percentage errors are less than one. Therefore, we might claim that the equation and its solutions are correct. From Tables 3 and 4 and Fig. 4, it is seen that the acceleration increased for $5000 < \mathrm{Re}_t \leq 299372$, $\beta = 0.021942857$, $n = 7$ and also $5000 < \mathrm{Re}_t \leq 200000$, $\beta = 0.11552$, $n = 5$. But the acceleration suddenly goes to retardation for $299373 \leq \mathrm{Re}_t \leq 310564$ and zero for $310565 \leq \mathrm{Re}_t < \infty$ when $\beta = 0.021942857$, $n = 7$. In the other hand, the acceleration suddenly goes to retardation for $207600 \leq \mathrm{Re}_t \leq 216500$ and zero for $216500 < \mathrm{Re}_t < \infty$ when $\beta = 0.11552$, $n = 5$.

The order of the boundary layer thickness in this case is $(\mu_t x^{n-1}/\rho\, U_t)^{1/n}$, $n > 2$ for turbulent flow is used for approximate estimation which converts to the boundary layer thickness for laminar flow if $n = 2$. Therefore, Eq. (2.17) transforms to the Blasius equation for laminar flow (Rahman [7] and Cortell [6]) for $n = 2$.

6. Convergences analysis and validation

Since the solution Eq. (1.3) of Eq. (1.2) is well-known solution. Then the solution Eq. (1.2) is following the convergence rules, $\sum_{i=0}^{\infty}\left|\left(-\frac{1}{2}\right)^i \frac{B_i \sigma^{i+1}}{(3i+2)\,!}\eta^{3i+2}\right| < \infty$. Taking absolute value on both sides of Eq. (3.3), we attain

$$|u(\eta)| = \left|\sum_{i=0}^{\infty} (-k)^i \frac{B_i\,\sigma^{i+1}}{(3i+2)\,!}\eta^{3i+2}\right|$$

$$\leq \sum_{i=0}^{\infty}\left|(-k)^i \frac{B_i\,\sigma^{i+1}}{(3i+2)\,!}\eta^{3i+2}\right|$$

$$= \sum_{i=0}^{\infty}\left|(2k)^i\right|\left|\left(-\frac{1}{2}\right)^i \frac{B_i\,\sigma^{i+1}}{(3i+2)\,!}\eta^{3i+2}\right|$$

$$= \sum_{i=0}^{\infty}\left|(2k)^i\right|\left|\left(-\frac{1}{2}\right)^i \frac{B_i\,\sigma^{i+1}}{(3i+2)\,!}\eta^{3i+2}\right|$$

Thus, $|u(\eta)| < \infty$.

Table 3
Numerical and analytical results for the values of Re_t, $\beta = 0.021942857$ and $n = 7$.

Reynolds no. Re_t	The value of k or $\beta\,Re_t^{\frac{n-2}{n}}$ for $n = 7$ $\beta = 0.021942857$	Numerical solution of $u''(0)$
5000	9.6253	1.5399
10,000	15.792	1.9724
20,000	25.9094	2.5264
30,000	34.6128	2.9201
40,000	42.5088	3.2361
50,000	49.854	3.5045
60,000	56.7882	3.7403
70,000	63.3983	3.952
80,000	69.7429	4.145
90,000	75.8644	4.3231
100,000	81.7941	4.4889
110,000	87.5564	4.6443
120,000	93.1708	4.7909
130,000	98.6529	4.9298
140,000	104.0157	5.0621
150,000	109.2701	5.1883
160,000	114.4252	5.3093
170,000	119.4891	5.4255
180,000	124.4684	5.5374
190,000	129.3694	5.6454
200,000	134.1971	5.7498
210,000	138.9563	5.8508
220,000	143.6512	5.9488
230,000	148.2855	6.044
240,000	152.8625	6.1366
250,000	157.3854	6.2267
260,000	161.8569	6.3146
270,000	166.2795	6.4002
280,000	170.6555	6.4839
290,000	174.9870	6.5657
291,000	175.4178	6.5738
292,000	175.8482	6.5818
293,000	176.2781	6.5899
294,000	176.7077	6.5979
295,000	177.1368	6.6059
296,000	177.5655	6.6139
297,000	177.9938	6.6219
298,000	178.4216	6.6298
299,000	178.8491	6.6378
299,200	178.9345	6.6393
299,355	179.0007	6.6406
299,360	179.0029	6.6406
299,365	179.0050	6.6406
299,370	179.0071	6.6407
299,371	179.0076	6.6407
299,372	179.008	6.6407
299,373	179.0084	−0.7512
299,374	179.0089	−0.7512
299,375	179.0093	−0.7512
299,380	179.0114	−0.7512
299,600	179.1054	−0.7514
300,000	179.2761	−0.7518
310,000	183.5246	−0.7607
310,500	183.736	−0.7611
310,550	183.7571	−0.7611
310,560	183.7613	−0.7611
310,564	183.763	−0.7611
310,565	183.7634	0
310,700	183.8205	0
310,840	183.8797	0
310,850	183.8797	0
350,000	200.1436	0
400,000	220.1732	0

Table 4
Numerical and analytical results for the values of Re_t, $\beta = 0.11552$ and $n = 5$.

Reynolds no. Re_t	The value of k or $\beta \, Re_t^{\frac{n-2}{n}}$ for $n = 5$ $\beta = 0.11552$	Numerical solution of $u''(0)$
5000	19.1443	2.1717
10,000	29.0173	2.6737
20,000	43.982	3.2917
30,000	56.0957	3.7174
40,000	66.6643	4.0525
50,000	76.2148	4.3331
60,000	85.0253	4.5767
70,000	93.2644	4.7933
80,000	101.0441	4.9892
90,000	108.4433	5.1687
100,000	115.52	5.3347
110,000	122.3187	5.4894
120,000	128.8742	5.6346
130,000	135.2145	5.7715
140,000	141.3624	5.9013
150,000	147.337	6.0247
160,000	153.1543	6.1425
170,000	158.8278	6.2552
180,000	164.3693	6.3634
190,000	169.7889	6.4674
200,000	175.0956	6.5677
210,000	180.2971	−0.7539
220,000	185.4005	0
230,000	190.4118	0
240,000	195.3367	0
250,000	200.1802	0
260,000	204.9468	0
270,000	209.6406	0
280,000	214.2654	0
290,000	218.8245	0
291,000	219.2769	0
292,000	219.7288	0
293,000	220.1799	0
294,000	220.6305	0
295,000	221.0805	0
296,000	221.5298	0
297,000	221.9786	0
298,000	222.4267	0
299,000	222.8742	0
299,200	222.9637	0
299,355	223.033	0
299,360	223.0352	0
299,365	223.0375	0
299,370	223.0397	0
299,371	223.0419	0
299,372	223.0442	0
299,373	223.1425	0
299,374	223.3212	0
299,375	227.7583	0
299,380	228.1328	0
299,600	223.3212	0
300,000	244.9616	0
310,000	265.3953	0
310,500	19.1443	0
310,550	29.0173	0
310,560	43.982	0
310,564	56.0957	0
310,565	66.6643	0
310,700	76.2148	0
310,850	93.2644	0
350,000	101.0441	0
400,000	108.4433	0

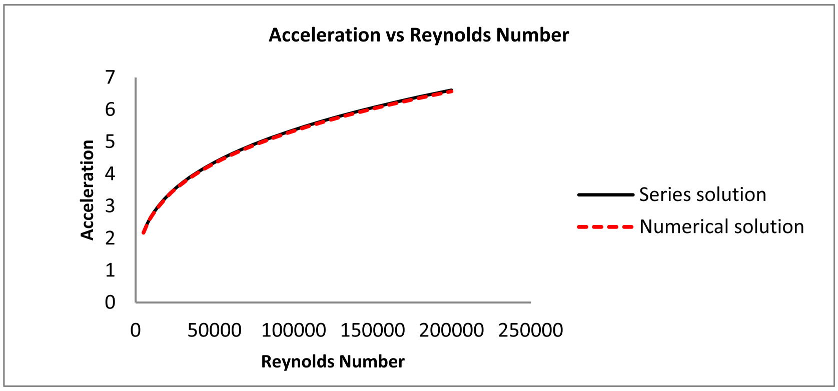

Fig. 2. Acceleration vs Reynolds number to increase acceleration how effect Reynolds number for $\beta = 0.11552$ and $n = 5$.

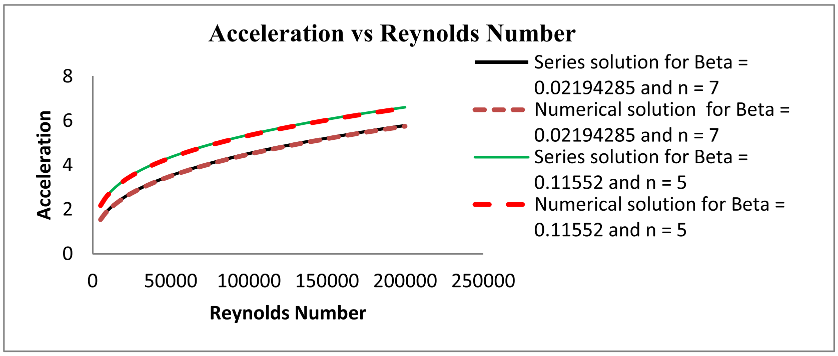

Fig. 3. Acceleration vs Reynolds number to increase acceleration how effect Reynolds number.

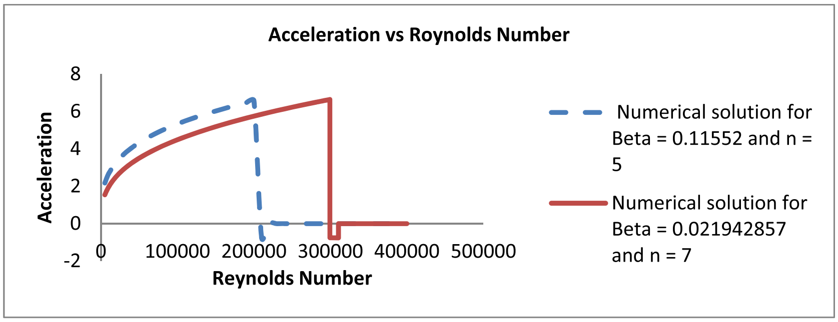

Fig. 4. Acceleration vs Reynolds number to increase acceleration how effect Reynolds number.

Since $\sum_{i=0}^{\infty}\left|\left(-\frac{1}{2}\right)^{i}\frac{B_i\sigma^{i+1}}{(3i+2)\,!}\eta^{3i+2}\right| < \infty$ and $\left|(2k)^{i}\right|$ is finite for $0 < k < \infty$. Therefore we obtain $|u(\eta)| < \infty$. Therefore, the solution converges.

Since the solution (3.3) converges and errors of numerical and approximate solutions shown in Tables 1 and 2 are less than one percent, therefore the obtained solution is effective.

7. Conclusion

In this article, a new and general nonlinear Blasius equation applicable to turbulent flow as well as laminar flow has been established, and the analytical approximate solutions and the numerical solutions through the finite difference technique using MATLAB have been examined. Tables 1–4 exhibit the results $u''(0)$ established in this study for several Reynolds number and specific value of β. From Figs. 1 and 2, it is observed that the acceleration is increasing when the Reynolds number is increased. Fig. 4 shows that the acceleration increased for $5000 < \mathrm{Re}_t \leq 299372$, $\beta = 0.021942857$ and $n = 7$, but acceleration rapidly goes to retardation for

$299373 \le Re_t \le 310564$ and zero for $310565 \le Re_t < \infty$. Thus, it is established that acceleration varies with Reynolds number for turbulent, constant pressure, steady and an incompressible flow.

Author contribution statement

M. Mizanur Rahman: Conceived and designed the experiments; Contributed reagents, materials, analysis tools, or data; Wrote the paper. Shahansha Khan: Performed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools, or data. M. Ali Akbar: Analyzed and interpreted the data; Contributed reagents, materials, analysis tools, or data.

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Data availability statement

No data was used for the research described in the article.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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