

Research Article

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Chaotic behaviors, stability, and solitary wave propagations of M-fractional LWE equation in magneto-electro-elastic circular rod

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Abstract: This work studies the chaotic behaviors and solitary wave propagations for the M-fractional longitudinal wave equation (M-fLWE). Here, we explain some assertions of the M-fractional derivative. Initially, we employ bifurcation theory to examine the chaotic behaviors that arise from the incorporation of diverse perturbation terms. We depict the phase portraits using three-dimensional (3D) and two-dimensional (2D) representations, Poincaré diagrams, and time-series plots. Furthermore, we utilize an enhanced modified F-expansion method to examine ion acoustic waves in the fLWE. The derived solutions manifest as trigonometric, exponential, and hyperbolic functions. In the numerical discussion, we present novel phenomena not observed in previous studies. For particular values of the free parameters, we discern luminous and obscure bell-shaped waves, periodic waves, periodic bell-shaped rogue waves, periodic rogue waves featuring singular solitons, periodic rogue waves, and interactions between periodic rogue waves and kink-shaped formations. Additionally, we juxtapose our results with the current literature to emphasize unique attributes in 2D, 3D, and density-based representations. This research provides significant insights into the intricate behaviors and varied waveforms of the governing model *via* a thorough investigation. This study enhances the comprehension of real-world physical phenomena through the examination of waveform attributes, bifurcation analysis, chaotic dynamics, and solitary waves.

Keywords: M-fractional derivative, electromagnetic, longitudinal waves, extended modified F-expansion technique

1 Introduction

Nonlinear evolution equations (NLEEs) play a pivotal role in understanding and modeling the dynamics of complex systems across various scientific and engineering disciplines [1–10]. Unlike linear equations, NLEEs can describe interactions where the effect is not directly proportional to the cause, thus capturing the essence of many natural and engineered processes. This nonlinearity is fundamental in explaining phenomena such as solitons in fiber optics, rogue waves in the ocean, and shock waves in aerodynamics. One of the most significant aspects of NLEEs is their ability to model systems that exhibit chaotic behavior. Chaos theory, a branch of mathematics that studies the behavior of dynamical systems that are highly sensitive to initial conditions, often relies on NLEEs to predict and understand chaotic systems. This has profound implications in fields such as weather forecasting, where small changes in initial conditions can lead to vastly different outcomes. In fluid dynamics, NLEEs such as the Navier–Stokes equations describe the flow of incompressible fluids. Solving these equations is critical for understanding turbulence, which has applications in everything from aeronautical engineering to environmental science. In plasma physics, NLEEs help in understanding the behavior of ionized gases, which is essential for advancements in fusion energy research.

The significance of NLEEs extends to the development of numerical methods and computational techniques. The solutions to NLEEs often exhibit rich structures and patterns, providing insights into the underlying physical processes. Techniques for analyzing NLEEs, such as modified approach [11], modified $\frac{G'}{G}$ -expansion techniques [12], homogeneous balance method [13], new extended auxiliary equation method [14], SE technique [15], Jacobi elliptic function method [16], Hirota technique [17], tanh method [18], Wronskian's technique [19], enhanced modified simple equation method technique [20,21], unified technique [22,23], generalized exponential rational function (GERF) method, and the generalized

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Kudryashov method [24], $\exp(-\phi(\xi))$ -expansion, and new form of modified Kudryashov's schemes [25–30], have advanced significantly, leading to practical applications in engineering, meteorology, and materials science. Their ability to describe real-world systems with high accuracy makes NLEEs indispensable in both theoretical research and practical problem-solving.

The propagation of longitudinal waves in a cylindrical rod made of a material with elastic, electromagnetic, and magneto-electric properties is governed by the longitudinal wave equation [31,32] in magneto-electro-elastic circular rod. This equation is based on the core principles of continuum mechanics and electromagnetism. The main objective of this work is to present multiple invariant solitary wave solutions for the time Mf-LWE:

$$D_{M,t}^{2\Theta,N} Q - l^2 Q_{xx} - \left(\frac{l^2}{2} Q^2 + m D_{M,t}^{2\Theta,N} Q \right)_{xx} = 0, \quad (1)$$

where $D_{M,t}^{2\Theta,N}$ is the M-fractional operator, Q represents the wave function, while l and m are the constants related to the material properties of the medium. The term $l^2 Q_{xx}$ represents the spatial variation of the wave, with Q_{xx} being the second spatial derivative of Q , indicating how the wave's shape changes along the rod. The expression $\left(\frac{l^2}{2} Q^2 + m D_{M,t}^{2\Theta,N} Q \right)_{xx}$ combines both nonlinear and additional derivative effects. The term $\frac{l^2}{2} Q^2$ introduces nonlinearity, which can describe phenomena such as wave steepening or the formation of solitary waves. $\exp(-\phi(\xi))$ expansion function process [31], the extended trial equation process [32], auxiliary equation mapping and direct algebraic process [33], Bernoulli sub-equation technique [34], mapping method [35], modified (P/P)-expansion method [36], improved modified extended tanh-function method [37], sine-Gordon process [38], ansatz functions method [39], ansatz method [40], unified method [41], GERF method [42], Super convergent finite element technique [43] dual $\left(\frac{G'}{G}, \frac{1}{G} \right)$ -expansion method [44], modified auxiliary equation method technique and Adomian decomposition method technique [45], unified Riccati equation expansion (UREE) and new Kudryashov method process [46], the extended sinh-Gordon equation expansion technique [47], harmonic balance technique [48], improved modified extended tanh function method [49], GERF technique [50], modified Sadar sub-equation method technique [51], Φ^6 -model expansion method [52], and so on [53,54]. The motivation of this work is to examine the chaotic nature and solitary wave solution of the M-fractional longitudinal wave equation (M-fLWE) equation. To obtain the solitary wave solution, we use a novel modified F-expansion technique and also analyze the

influence of the M-fractional operator on the obtained solutions. Additionally, we do a comprehensive investigation of the chaotic behaviors resulting from the introduction of various perturbation terms for the first time. The examination of chaos in these systems is crucial for comprehending the shift from regular to irregular wave dynamics, which holds considerable significance for disciplines such as plasma physics, nonlinear optics, and fluid mechanics. To thoroughly illustrate these dynamics, we create phase portraits utilizing three-dimensional (3D) and two-dimensional (2D) representations, Poincaré diagrams, and time-series plots. These graphical representations not only corroborate our analytical results but also offer profound insights into the complex dynamics of the system.

The organization of this work is as follows:

Section 2 discusses the characteristics and properties of the truncated M-fractional derivative, along with the extended modified F-expansion technique.

Section 3 explores the mathematical analysis of the M-fractional longitudinal wave equation (LWE) equation.

Section 4 examines the chaotic behaviors and solitary wave propagation in the M-fractional LWE equation.

Section 5 focuses on applying the extended modified F-expansion technique.

Section 6 presents and analyzes graphs, emphasizing the effects of the fractional parameter.

Section 7 compares this research with other published studies.

Finally, Section 8 summarizes the conclusions.

2 Methodology

2.1 M-fractional derivative

Definition: consider $\chi : (0, \infty) \rightarrow \mathbb{R}$ the TMD of χ with order n exhibit as

$$D_{M,t}^{\Theta,N} \chi(t) = \lim_{h \rightarrow 0} \frac{\chi(t H_n(h t^{1-\Theta})) - \chi(t)}{h}; \quad 0 < \Theta < 1, \quad N > 0, \text{ where}$$

$H_n(\cdot)$ is a truncated Mittag-Leffler function of one parameter that defined as [47]

$$H_n(z) = \sum_{j=0}^n \frac{z^j}{\Gamma(nj+1)}.$$

Characteristics:

Suppose $0 < \Theta < 1$, $N > 0$, $a, b \in \mathfrak{R}$ and χ, ϕ , Θ -differentiable at a point $t > 0$, then

$$(1) \quad D_{M,t}^{\Theta,N} (a\chi(t) + b\phi(t)) = a D_{M,t}^{\Theta,N} \chi(t) + b D_{M,t}^{\Theta,N} \phi(t).$$

$$(2) \quad D_{M,t}^{\Theta,N} (\chi(t)\phi(t)) = \chi(t) D_{M,t}^{\Theta,N} \phi(t) + \phi(t) D_{M,t}^{\Theta,N} \chi(t).$$

$$(3) \quad D_{M,t}^{\Theta,N} \left(\frac{\chi(t)}{\phi(t)} \right) = \frac{\chi(t) D_{M,t}^{\Theta,N} \phi(t) + \phi(t) D_{M,t}^{\Theta,N} \chi(t)}{\phi(t)^2}.$$

$$(4) \quad D_{M,t}^{\theta,N}(c) = 0, \text{ where } \chi(t) = c.$$

$$(5) \quad D_{M,t}^{\theta,N}\chi(t) = \frac{t^{1-\theta}}{\Gamma(n+1)} \frac{d\chi(t)}{dt}.$$

2.2 Extended modified F-expansion method

In this section, we explain a new technique, namely, the extended modified F-expansion technique [55,56], which is the combination of the modified F-expansion technique and the modified Kudryashov technique. Consider the M-fractional nonlinear model:

$$\psi(D_{M,t}^{\mu,\gamma}Q, QQ_{xx}, D_{M,t}^{2\mu,\gamma}Q, Q_{xxx}) = 0, \quad (2)$$

where $D_{M,t}^{\mu,\gamma}$ is the M-fractional operator, $Q = Q(x, t)$ is the wave function, and ψ is the function Q and its partial derivative.

Step-01: At first, we use a transformation variable in the following form:

$$Q(x, t) = Q(\varphi); \quad \varphi = C_1x - v\frac{\Gamma(\gamma+1)}{\mu}t^\mu, \quad (3)$$

where C_1 is the wave frequency, v is the wave speed, and μ is the order of fractional derivative.

To convert Eq. (2) into the ordinary differential equation (ODE) form, we insert Eq. (3) into Eq. (2):

$$\psi_1(C_1vQ_{\varphi\varphi}, C_2QQ_{\varphi\varphi}, v^2Q_{\varphi\varphi\varphi}, C_2^3Q_{\varphi\varphi\varphi\varphi}) = 0, \quad (4)$$

where Eq. (4) is the ODE form of Eq. (2).

Now, consider a trial solution of Eq. (4) in the following form [57]:

$$\chi = \frac{\sum_{i=0}^n a_i(H(\varphi))^i}{\sum_{j=0}^m b_j(H(\varphi))^j}; \quad a_n, b_m \neq 0, \quad (5)$$

where m , and n are the integer numbers. By balancing the highest-order derivative and nonlinear term in Eq. (4), we obtain the values of m , and n by using the following formula:

$$\begin{aligned} \frac{d^p Q}{d\varphi^p} &= (n-m) + P; \\ Q^q \left(\frac{d^p Q}{d\varphi^p} \right)^s &= q(n-m) + g((n-m) + P), \end{aligned}$$

where $H(\varphi)$ satisfied the auxiliary equation [58],

$$\frac{dH(\varphi)}{d\varphi} = R_1 + R_2H(\varphi) + R_3H(\varphi)^2. \quad (6)$$

Insert Eq. (5) with Eq. (6) into Eq. (4), then we achieve a system of equations after considering the co-efficient of

$(\mathcal{G}(\theta))^{\mp i}$ equal to zero. If we solve the obtained system of equations, then we obtain the destinations of Eq. (3):

$$R_1 = 0, R_2 = 1, R_3 = -1; H(\varphi) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{\varphi}{2}\right).$$

For

$$R_1 = 0, R_2 = 1, R_3 = -1; H(\varphi) = \frac{1}{2} - \frac{1}{2}\coth\left(\frac{\varphi}{2}\right).$$

For

$$R_1 = \frac{1}{2}, R_2 = 0, R_3 = -\frac{1}{2}; H(\varphi) = \coth(\varphi) \pm \operatorname{cosech}(\varphi).$$

For

$$R_1 = 1, R_2 = 0, R_3 = -1; H(\varphi) = \tanh(\varphi), \coth(\varphi).$$

For

$$R_1 = \frac{1}{2}, R_2 = 0, R_3 = \frac{1}{2}; H(\varphi) = \sec(\varphi) + \tan(\varphi).$$

For

$$R_1 = -\frac{1}{2}, R_2 = 0, R_3 = -\frac{1}{2}; H(\varphi) = \sec(\varphi) - \tan(\varphi).$$

For

$$R_1 = R_2 = 0; H(\varphi) = \frac{1}{R_3\varphi + \delta}.$$

For

$$R_2 = R_3 = 0; H(\varphi) = R_1\varphi.$$

For

$$R_1 = 0, R_2 > 0, R_3 < 0; H(\varphi) = \frac{R_2 \exp(R_2(\varphi + \delta))}{1 - R_3 \exp(R_2(\varphi + \delta))}.$$

For

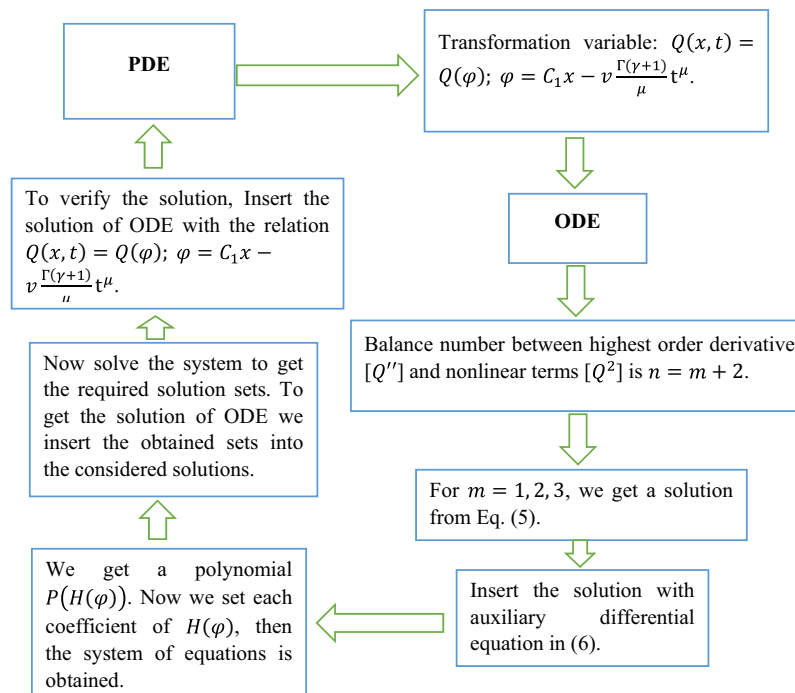
$$R_1 = 0, R_2 < 0, R_3 > 0; H(\varphi) = -\frac{R_2 \exp(R_2(\varphi + \delta))}{1 + R_3 \exp(R_2(\varphi + \delta))}.$$

For

$$R_3 = 0; H(\varphi) = \frac{-R_1 + e^{R_2\varphi}}{R_2}.$$

2.3 Flow chart of extended modified F-expansion method

In this subsection, we make a flow chart of the extended modified F-expansion (EMFE) method to solve the time M-fractional longitudinal wave equation.



3 Mathematical analysis

The time M-fLWE equation can be written as

$$D_{M,t}^{2\Theta,N} Q - l^2 Q_{xx} - \left(\frac{l^2}{2} Q^2 + m D_{M,t}^{2\Theta,N} Q \right)_{xx} = 0. \quad (7)$$

Consider the relation

$$\varphi = kx - \lambda \frac{\Gamma(N+1)t^\Theta}{\Theta}; H(x, t) = H(\varphi). \quad (8)$$

If we substitute Eq. (8) into Eq. (7), then the obtained form is

$$k^2 \lambda^2 m Q^{iv} + (l^2 k^2 - \lambda^2) Q'' + \frac{l^2 k^2}{2} (Q^2)'' = 0. \quad (9)$$

Now we integrate Eq. (9), and then we obtain

$$k^2 \lambda^2 m Q''' + (l^2 k^2 - \lambda^2) Q' + \frac{l^2 k^2}{2} Q^2 = 0. \quad (10)$$

4 Chaotic analysis

Partial differential equation solutions can have chaotic dynamics due to bifurcations, which appear random and complex. In the preceding section, chaos was absent in the planar dynamical structure (10). However, the chaotic feature is only observed when an external perturbation term is included. This perturbed form of the planar dynamical structure (10) can be expressed as follows:

$$\begin{cases} \frac{dQ}{d\varphi} = G, \\ \frac{dG}{d\varphi} = \frac{-(l^2 k^2 - \lambda^2) Q - \frac{l^2 k^2}{2} Q^2}{k^2 \lambda^2 m} + P(\varphi), \end{cases} \quad (11)$$

where $P(\varphi)$ is the added perturbation terms. Following are some perturbation terms that can be used to explain chaotic nature:

Trigonometric form: If $P(\varphi) = 1.5 \cos(3.9\varphi)$, then Figure 1 shows the phase portrait of Eq. (11), and Figure 2 shows the multistability of Eq. (11) for $k = \lambda = m = 1$, and $l = 2$.

Gaussian form: If $P(\varphi) = 3.5 \exp\left(-\frac{(0.12\varphi)^2}{2}\right)$, then Figure 3 shows the phase portrait of Eq. (11) and Figure 4 shows the multistability of Eq. (11).

Hyperbolic form: If $P(\varphi) = 1.4 \cosh(0.04\varphi)$, then Figure 5 shows the phase portrait of Eq. (11) and Figure 6 shows the multistability of Eq. (11) for $k = \lambda = m = 1$, and $l = 2$.

5 Solitary wave solution of M-fractional LW model

In this section, we implement the EMFE technique to investigate the solitary wave pattern of the time M-fLWE model.

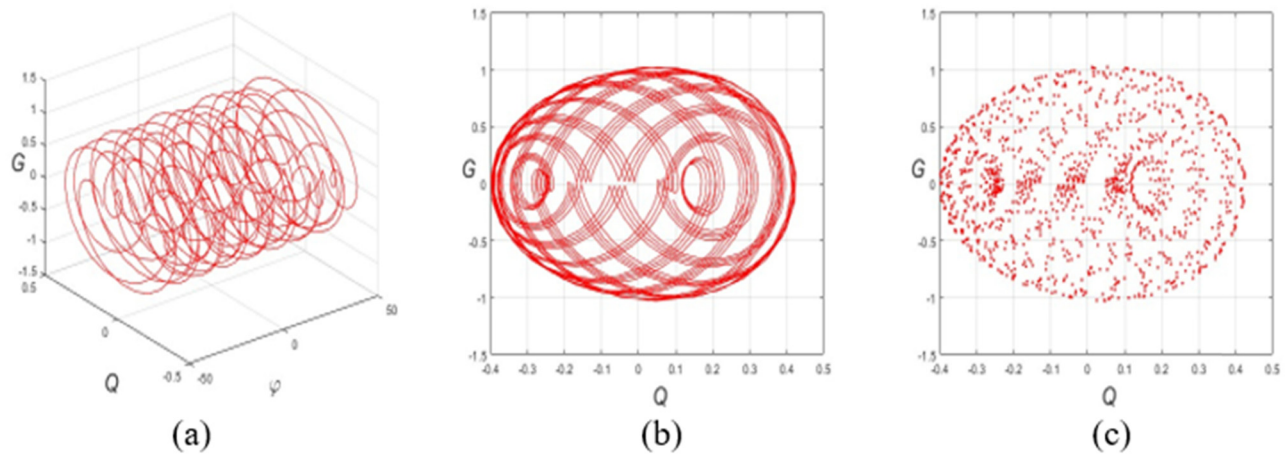


Figure 1: Diagram of the perturbed form Eq. (11) for $P(\varphi) = 1.5\cos(3.9\varphi)$ with beginning condition $(Q(0), G(0)) = (0.1, 0.1)$. (a) 3D phase shape. (b) 2D phase shape. (c) Poincaré plot.

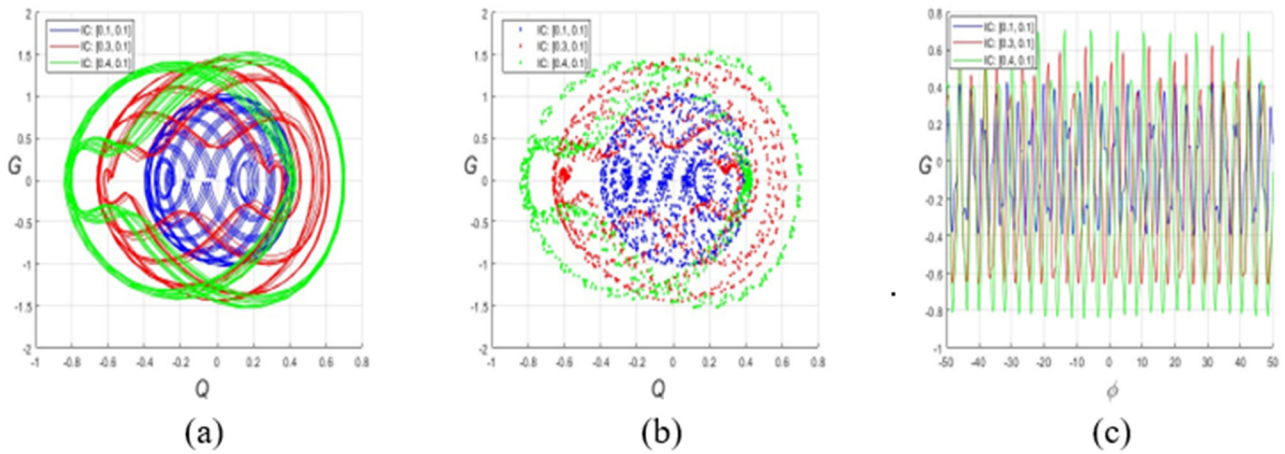


Figure 2: Multistability of the perturbed form Eq. (11) with $P(\varphi) = 1.5\cos(3.9\varphi)$. (a) 2D phase shape. (b) Poincaré plot. (c) Time-series diagram.

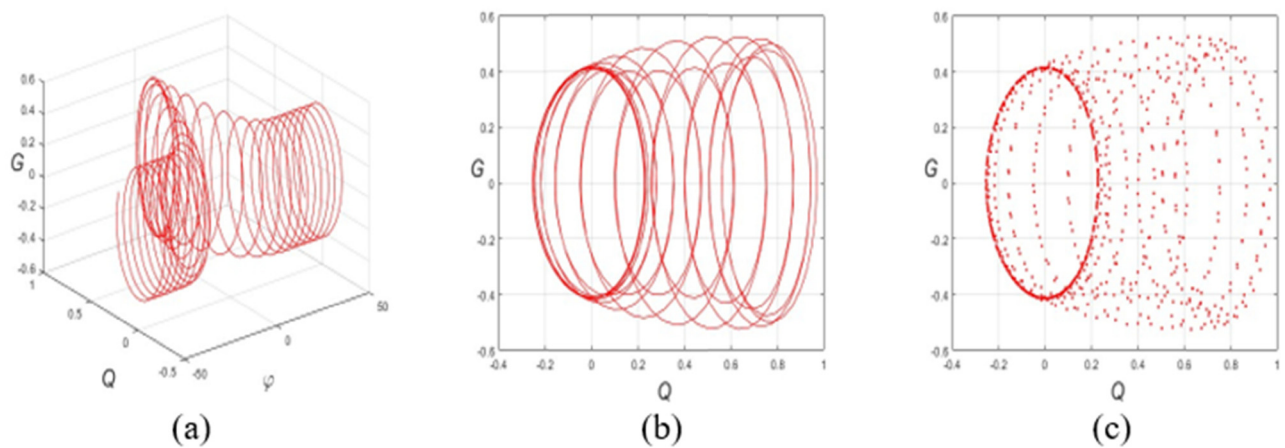


Figure 3: Diagram of the perturbed form Eq. (11) for $P(\varphi) = 3.5\exp\left(-\frac{(0.12\varphi)^2}{2}\right)$ with beginning condition $(Q(0), G(0)) = (0.2, 0.2)$. (a) 3D phase shape. (b) 2D phase shape. (c) Poincaré plot.

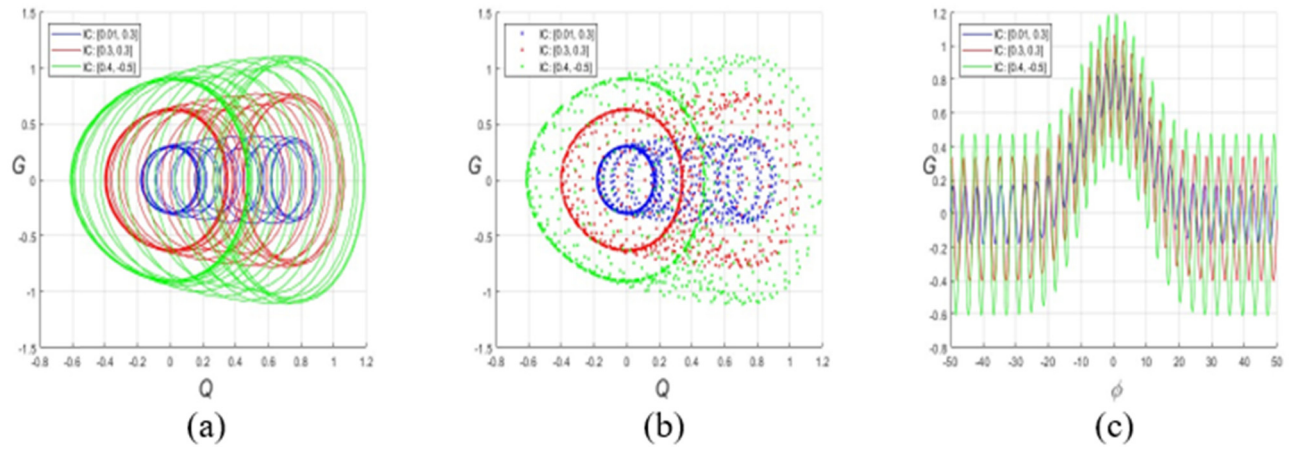


Figure 4: Multistability of the perturbed form Eq. (11) with $P(\varphi) = 3.5\exp\left(-\frac{(0.12\varphi)^2}{2}\right)$. (a) 2D phase shape. (b) Poincare plot. (c) Time-series diagram.

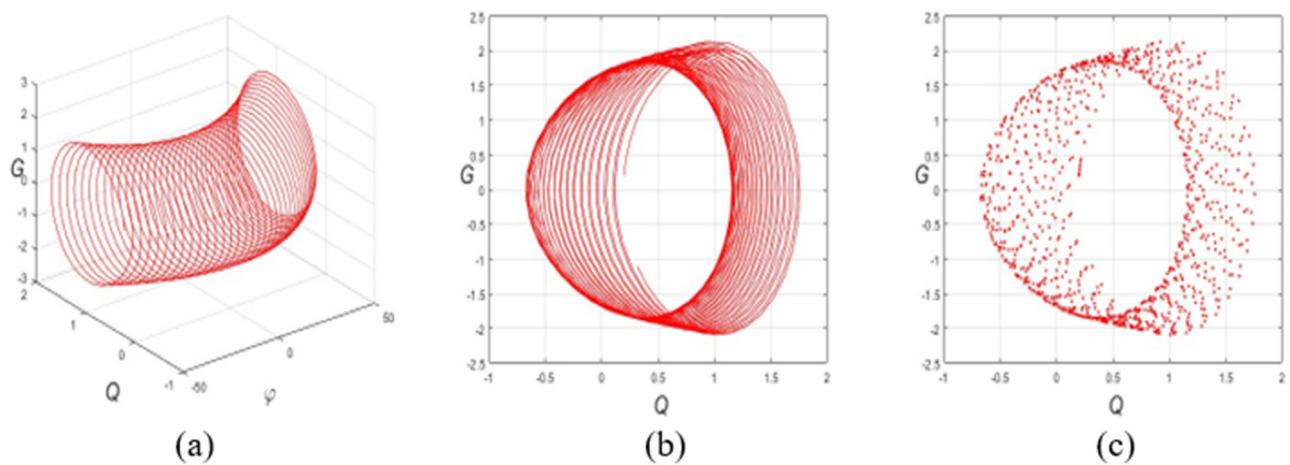


Figure 5: Diagram of the perturbed form Eq. (11) for $P(\varphi) = 1.4\cosh(0.04\varphi)$ with beginning condition $(Q(0), G(0)) = (0.2, 0.2)$. (a) 3D phase shape. (b) 2D phase shape. (c) Poincare plot.

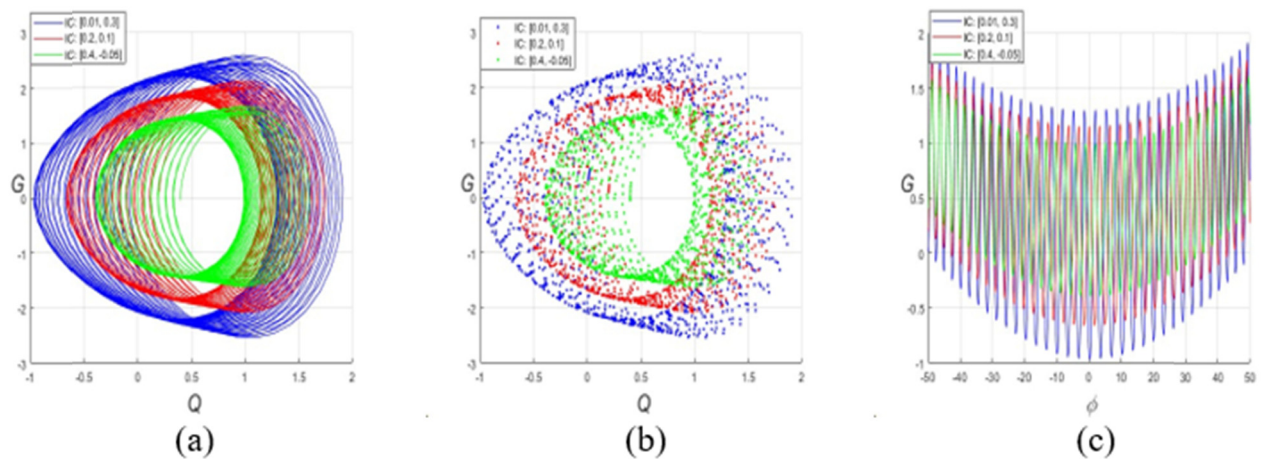


Figure 6: Multistability of the perturbed form Eq. (11) with $P(\varphi) = 1.4\cosh(0.04\varphi)$. (a) 2D phase shape. (b) Poincare plot. (c) Time-series diagram.

According to the EMFE technique, the trial solution is the following form:

$$Q(\varphi) = \frac{\beta_0 + \beta_1 H(\varphi) + \beta_2 H(\varphi)^2 + \beta_3 H(\varphi)^3}{\gamma_0 + \gamma_1 H(\varphi)}. \quad (12)$$

If we substitute Eq. (12) and Eq. (6) into Eq. (10), then the following solution sets obtain:

$$k = \sqrt{\frac{-\beta_3}{(4m\beta_3 R_1 R_3 - m\beta_3 R_2^2 + 12m\gamma_1 R_3^2)}}, \quad \lambda = \frac{l}{R_3} \sqrt{\frac{-\beta_3}{12m\gamma_1}}, \quad \beta_0 = \frac{(\beta_2 R_3 - \beta_3 R_2) R_1}{R_3^2}, \quad \beta_1 = \frac{\beta_2 R_2 R_3 + \beta_3 R_1 R_3 - \beta_3 R_2^2}{R_3^2},$$

$$\gamma_0 = \frac{\gamma_1 (\beta_2 R_3 - \beta_3 R_2)}{\beta_3 R_3}.$$

$$k = \sqrt{\frac{-1}{(4\lambda^2 m R_1 R_3 - \lambda^2 m R_2^2 - l^2)}}, \quad \lambda, \beta_0 = -\frac{12\lambda^2 m R_1 R_3 \gamma_0}{l^2}, \quad \beta_1 = -\frac{12\lambda^2 m R_3 (\gamma_0 R_2 + \gamma_1 R_1)}{l^2}, \quad \beta_2 = -\frac{12\lambda^2 m R_3 (\gamma_0 R_3 + \gamma_1 R_2)}{l^2},$$

$$\beta_3 = -\frac{12\lambda^2 m \gamma_1 R_3^2}{l^2}.$$

$$\text{Case 01: } k = \sqrt{\frac{-\beta_3}{(4m\beta_3 R_1 R_3 - m\beta_3 R_2^2 + 12m\gamma_1 R_3^2)}}, \quad \lambda = \frac{l}{R_3} \sqrt{\frac{-\beta_3}{12m\gamma_1}}, \quad \beta_0 = \frac{(\beta_2 R_3 - \beta_3 R_2) R_1}{R_3^2}, \quad \beta_1 = \frac{\beta_2 R_2 R_3 + \beta_3 R_1 R_3 - \beta_3 R_2^2}{R_3^2}, \quad \gamma_0 = \frac{\gamma_1 (\beta_2 R_3 - \beta_3 R_2)}{\beta_3 R_3}.$$

For, $R_1 = 0$, $R_2 = 1$, $R_3 = -1$;

$$Q(x, t) = \frac{\beta_0 + \beta_1 \left(\frac{1}{2} + \frac{1}{2} \tanh\left(\frac{\varphi}{2}\right) \right) + \beta_2 \left(\frac{1}{2} + \frac{1}{2} \tanh\left(\frac{\varphi}{2}\right) \right)^2 + \beta_3 \left(\frac{1}{2} + \frac{1}{2} \tanh\left(\frac{\varphi}{2}\right) \right)^3}{\gamma_0 + \gamma_1 \left(\frac{1}{2} + \frac{1}{2} \tanh\left(\frac{\varphi}{2}\right) \right)}. \quad (13)$$

For $R_1 = 0$, $R_2 = 1$, $R_3 = -1$.

$$Q(x, t) = \frac{\beta_0 + \beta_1 \left(\frac{1}{2} - \frac{1}{2} \coth\left(\frac{\varphi}{2}\right) \right) + \beta_2 \left(\frac{1}{2} - \frac{1}{2} \coth\left(\frac{\varphi}{2}\right) \right)^2 + \beta_3 \left(\frac{1}{2} - \frac{1}{2} \coth\left(\frac{\varphi}{2}\right) \right)^3}{\gamma_0 + \gamma_1 \left(\frac{1}{2} - \frac{1}{2} \coth\left(\frac{\varphi}{2}\right) \right)}. \quad (14)$$

For $R_1 = \frac{1}{2}$, $R_2 = 0$, $R_3 = -\frac{1}{2}$.

$$Q(x, t) = \frac{\beta_0 + \beta_1 (\coth(\varphi) \pm \operatorname{cosch}(\varphi)) + \beta_2 (\coth(\varphi) \pm \operatorname{cosch}(\varphi))^2 + \beta_3 (\coth(\varphi) \pm \operatorname{cosch}(\varphi))^3}{\gamma_0 + \gamma_1 (\coth(\varphi) \pm \operatorname{cosch}(\varphi))}. \quad (15)$$

For $R_1 = 1$, $R_2 = 0$, $R_3 = -1$

$$Q(x, t) = \frac{\beta_0 + \beta_1 (\tanh(\varphi),) + \beta_2 (\tanh(\varphi),)^2 + \beta_3 (\tanh(\varphi),)^3}{\gamma_0 + \gamma_1 (\tanh(\varphi),)}. \quad (16)$$

For $R_1 = \frac{1}{2}$, $R_2 = 0$, $R_3 = \frac{1}{2}$.

$$Q(x, t) = \frac{\beta_0 + \beta_1 (\sec(\varphi) + \tan(\varphi)) + \beta_2 (\sec(\varphi) + \tan(\varphi))^2 + \beta_3 (\sec(\varphi) + \tan(\varphi))^3}{\gamma_0 + \gamma_1 (\sec(\varphi) + \tan(\varphi))}. \quad (17)$$

For $R_1 = -\frac{1}{2}$, $R_2 = 0$, $R_3 = -\frac{1}{2}$.

$$Q(x, t) = \frac{\beta_0 + \beta_1 (\sec(\varphi) - \tan(\varphi)) + \beta_2 (\sec(\varphi) - \tan(\varphi))^2 + \beta_3 (\sec(\varphi) - \tan(\varphi))^3}{\gamma_0 + \gamma_1 (\sec(\varphi) - \tan(\varphi))}. \quad (18)$$

For $R_1 = R_2 = 0$; $H(\varphi) = \frac{1}{R_3 \varphi + \delta}$.

$$Q(x, t) = \frac{\beta_0 + \beta_1 \left(\frac{1}{R_3 \varphi + \delta} \right) + \beta_2 \left(\frac{1}{R_3 \varphi + \delta} \right)^2 + \beta_3 \left(\frac{1}{R_3 \varphi + \delta} \right)^3}{\gamma_0 + \gamma_1 \left(\frac{1}{R_3 \varphi + \delta} \right)}. \quad (19)$$

For $R_2 = R_3 = 0$.

$$Q(x, t) = \frac{\beta_0 + \beta_1(R_1\varphi) + \beta_2(R_1\varphi)^2 + \beta_3(R_1\varphi)^3}{\gamma_0 + \gamma_1(R_1\varphi)}. \quad (20)$$

For $R_1 = 0$, $R_2 > 0$, $R_3 < 0$.

$$Q(x, t) = \frac{\beta_0 + \beta_1 \left(\frac{R_2 \exp(R_2(\varphi + \delta))}{1 - R_3 \exp(R_2(\varphi + \delta))} \right) + \beta_2 \left(\frac{R_2 \exp(R_2(\varphi + \delta))}{1 - R_3 \exp(R_2(\varphi + \delta))} \right)^2 + \beta_3 \left(\frac{R_2 \exp(R_2(\varphi + \delta))}{1 - R_3 \exp(R_2(\varphi + \delta))} \right)^3}{\gamma_0 + \gamma_1 \left(\frac{R_2 \exp(R_2(\varphi + \delta))}{1 - R_3 \exp(R_2(\varphi + \delta))} \right)}. \quad (21)$$

For $R_1 = 0$, $R_2 < 0$, $R_3 > 0$.

$$Q(x, t) = \frac{\beta_0 + \beta_1 \left(\frac{-R_2 \exp(R_2(\varphi + \delta))}{1 + R_3 \exp(R_2(\varphi + \delta))} \right) + \beta_2 \left(\frac{-R_2 \exp(R_2(\varphi + \delta))}{1 + R_3 \exp(R_2(\varphi + \delta))} \right)^2 + \beta_3 \left(\frac{-R_2 \exp(R_2(\varphi + \delta))}{1 + R_3 \exp(R_2(\varphi + \delta))} \right)^3}{\gamma_0 + \gamma_1 \left(\frac{-R_2 \exp(R_2(\varphi + \delta))}{1 + R_3 \exp(R_2(\varphi + \delta))} \right)}, \quad (22)$$

$$\text{where } \varphi = \sqrt{\frac{-\beta_3}{(4m\beta_3 R_1 R_3 - m\beta_3 R_2^2 + 12m\gamma_1 R_3^2)}} \times -\frac{l}{R_3} \sqrt{\frac{-\beta_3}{12m\gamma_1}} \frac{\Gamma(N+1)t^\Theta}{\Theta}$$

for, $R_3 = 0$.

The solution is rejected.

$$\text{Case 02: } k = \sqrt{\frac{-1}{(4\lambda^2 m R_1 R_3 - \lambda^2 m R_2^2 - l^2)}} \lambda, \beta_0 = -\frac{12\lambda^2 m R_1 R_3 \gamma_0}{l^2}, \beta_1 = -\frac{12\lambda^2 m R_3 (\gamma_0 R_2 + \gamma_1 R_1)}{l^2}, \beta_2 = -\frac{12\lambda^2 m R_3 (\gamma_0 R_3 + \gamma_1 R_2)}{l^2}, \beta_3 = -\frac{12\lambda^2 m \gamma_1 R_3^2}{l^2}.$$

For $R_1 = 0$, $R_2 = 1$, $R_3 = -1$;

$$Q(x, t) = \frac{\beta_1 \left(\frac{1}{2} + \frac{1}{2} \tanh\left(\frac{\varphi}{2}\right) \right) + \beta_2 \left(\frac{1}{2} + \frac{1}{2} \tanh\left(\frac{\varphi}{2}\right) \right)^2 + \beta_3 \left(\frac{1}{2} + \frac{1}{2} \tanh\left(\frac{\varphi}{2}\right) \right)^3}{\gamma_0 + \gamma_1 \left(\frac{1}{2} + \frac{1}{2} \tanh\left(\frac{\varphi}{2}\right) \right)}. \quad (23)$$

For $R_1 = 0$, $R_2 = 1$, $R_3 = -1$.

$$Q(x, t) = \frac{\beta_1 \left(\frac{1}{2} - \frac{1}{2} \coth\left(\frac{\varphi}{2}\right) \right) + \beta_2 \left(\frac{1}{2} - \frac{1}{2} \coth\left(\frac{\varphi}{2}\right) \right)^2 + \beta_3 \left(\frac{1}{2} - \frac{1}{2} \coth\left(\frac{\varphi}{2}\right) \right)^3}{\gamma_0 + \gamma_1 \left(\frac{1}{2} - \frac{1}{2} \coth\left(\frac{\varphi}{2}\right) \right)}. \quad (24)$$

For $R_1 = \frac{1}{2}$, $R_2 = 0$, $R_3 = -\frac{1}{2}$.

$$Q(x, t) = \frac{\beta_0 + \beta_1(\coth(\varphi) \pm \operatorname{cosech}(\varphi)) + \beta_2(\coth(\varphi) \pm \operatorname{cosech}(\varphi))^2 + \beta_3(\coth(\varphi) \pm \operatorname{cosech}(\varphi))^3}{\gamma_0 + \gamma_1(\coth(\varphi) \pm \operatorname{cosech}(\varphi))}. \quad (25)$$

For $R_1 = 1$, $R_2 = 0$, $R_3 = -1$

$$Q(x, t) = \frac{\beta_0 + \beta_1(\tanh(\varphi),) + \beta_2(\tanh(\varphi),)^2 + \beta_3(\tanh(\varphi),)^3}{\gamma_0 + \gamma_1(\tanh(\varphi),)}. \quad (26)$$

For $R_1 = \frac{1}{2}$, $R_2 = 0$, $R_3 = \frac{1}{2}$.

$$Q(x, t) = \frac{\beta_0 + \beta_1(\sec(\varphi) + \tan(\varphi)) + \beta_2(\sec(\varphi) + \tan(\varphi))^2 + \beta_3(\sec(\varphi) + \tan(\varphi))^3}{\gamma_0 + \gamma_1(\sec(\varphi) + \tan(\varphi))}. \quad (27)$$

For $R_1 = -\frac{1}{2}$, $R_2 = 0$, $R_3 = -\frac{1}{2}$.

$$Q(x, t) = \frac{\beta_0 + \beta_1(\sec(\varphi) - \tan(\varphi)) + \beta_2(\sec(\varphi) - \tan(\varphi))^2 + \beta_3(\sec(\varphi) - \tan(\varphi))^3}{\gamma_0 + \gamma_1(\sec(\varphi) - \tan(\varphi))}. \quad (28)$$

For $R_1 = R_2 = 0$; $H(\varphi) = \frac{1}{R_3\varphi + \delta}$.

$$Q(x, t) = \frac{\beta_1 \left(\frac{1}{R_3 \varphi + \delta} \right) + \beta_2 \left(\frac{1}{R_3 \varphi + \delta} \right)^2 + \beta_3 \left(\frac{1}{R_3 \varphi + \delta} \right)^3}{\gamma_0 + \gamma_1 \left(\frac{1}{R_3 \varphi + \delta} \right)}. \quad (29)$$

For $R_2 = R_3 = 0$.

$$Q(x, t) = \frac{\beta_1(R_1 \varphi) + \beta_2(R_1 \varphi)^2}{\gamma_0 + \gamma_1(R_1 \varphi)}. \quad (30)$$

For $R_1 = 0$, $R_2 > 0$, $R_3 < 0$.

$$Q(x, t) = \frac{\beta_1 \left(\frac{R_2 \exp(R_2(\varphi + \delta))}{1 - R_3 \exp(R_2(\varphi + \delta))} \right) + \beta_2 \left(\frac{R_2 \exp(R_2(\varphi + \delta))}{1 - R_3 \exp(R_2(\varphi + \delta))} \right)^2 + \beta_3 \left(\frac{R_2 \exp(R_2(\varphi + \delta))}{1 - R_3 \exp(R_2(\varphi + \delta))} \right)^3}{\gamma_0 + \gamma_1 \left(\frac{R_2 \exp(R_2(\varphi + \delta))}{1 - R_3 \exp(R_2(\varphi + \delta))} \right)}. \quad (31)$$

For $R_1 = 0$, $R_2 < 0$, $R_3 > 0$.

$$Q(x, t) = \frac{\beta_1 \left(\frac{-R_2 \exp(R_2(\varphi + \delta))}{1 + R_3 \exp(R_2(\varphi + \delta))} \right) + \beta_2 \left(\frac{-R_2 \exp(R_2(\varphi + \delta))}{1 + R_3 \exp(R_2(\varphi + \delta))} \right)^2 + \beta_3 \left(\frac{-R_2 \exp(R_2(\varphi + \delta))}{1 + R_3 \exp(R_2(\varphi + \delta))} \right)^3}{\gamma_0 + \gamma_1 \left(\frac{-R_2 \exp(R_2(\varphi + \delta))}{1 + R_3 \exp(R_2(\varphi + \delta))} \right)}, \quad (32)$$

where $\varphi = \sqrt{\frac{-1}{(4\lambda^2 m R_1 R_3 - \lambda^2 m R_2^2 - l^2)}} \lambda x - \lambda \frac{\Gamma(N+1)t^\theta}{\theta}$.

For $R_3 = 0$.

The solution is rejected.

6 Figure analysis

In this section, we explore various wave phenomena of the M-fractional LWE equation using the extended modified F-expansion technique. Two-dimensional plots illustrate the impact of M-fractional parameters. The solutions derived in this study are crucial for advancing magneto-electro-elastic (MEE) materials and devices, which have important applications in areas such as vibration control,

structural health monitoring, and energy harvesting. Understanding the dynamics of these waves can enhance the efficiency and functionality of MEE systems. This research highlights the roles of different waveforms in affecting wave propagation and energy transfer within the material. Bright soliton is a localized energy pulse that preserves its form, essential for energy transmission, signal enhancement, and mechanical wave propagation in intelligent materials. Dark soliton is a localized depression in the wave field, advantageous for wave attenuation, energy flow regulation, and vibration mitigation in engineered MEE systems. Kink wave is a mobile transition between two equilibrium states, crucial for phase transformations, domain wall dynamics, and nonlinear material

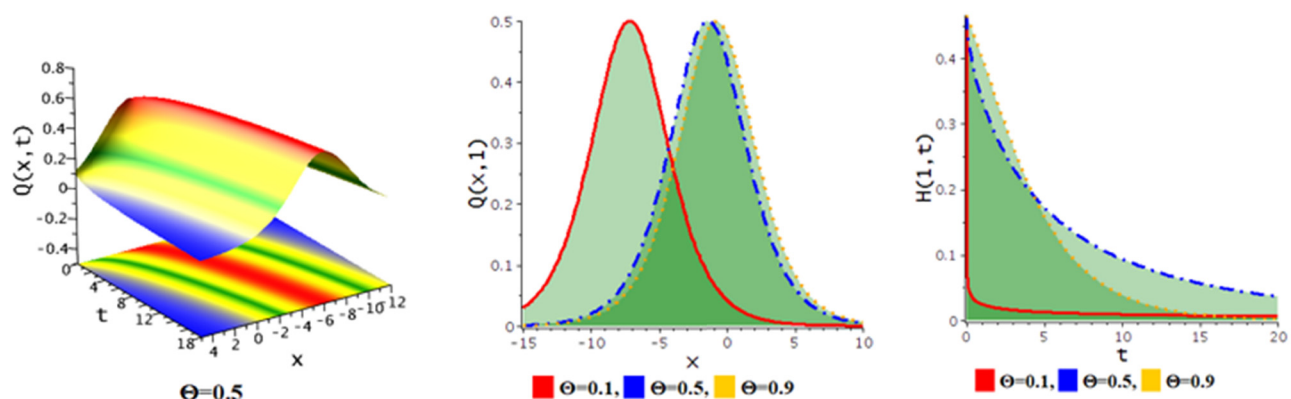


Figure 7: Diagram of bright bell soliton of the solution Eq. (13) for the values $m = 0.5$, $l = 0.5$, $\beta_3 = -1$, $\beta_2 = 0.1$, $\gamma_1 = 0.5$, $N = 1.5$.

responses. Periodic wave is a continuous oscillation pattern crucial for resonance analysis, vibration control, and wave-based sensing applications. These wave structures are fundamental in smart materials, energy harvesting, and wave-based communication systems, enhancing the understanding of nonlinear wave dynamics in coupled MEE environments. The complex phenomena are analyzed in real, imaginary, and absolute forms using 3D and 2D diagrams. In the 3D diagrams, we show the effect of fractional parameters for $\Theta = 0.5$ and compare it with the classical (original) form of the LWE model. In 2D diagram, we have shown the effect of M-fractional parameters for $\Theta = 0.1, 0.5, 0.9$. Figure 7 provides the diagram of bright bell-shaped for the solution Eq. (13) for the values $m = 0.5$, $l = 0.5$, $\beta_3 = -1$, $\beta_2 = 0.1$, $\gamma_1 = 0.5$, $N = 1.5$. For $\beta_3\gamma_1 < 0$ and $m > 0$, the solution Eq. (13) is a real-valued function and always represents bright solitary wave such as in Figure 7. Figure 8 represents the dark bell-shaped wave of the solution Eq. (13) for the values $m = -0.5$, $l = 0.5$, $\beta_3 = 1$, $\beta_2 = 0.1$, $\gamma_1 = 0.5$, $N = 1.5$. For $\beta_3\gamma_1 > 0$ and

$m < 0$, the solution Eq. (13) is a real-valued function and always represents the dark solitary wave such as in Figure 8. Figure 9 provides the periodic wave solutions of the solution Eq. (17) for the values $m = -0.5$, $l = 0.5$, $\beta_3 = -1$, $\beta_2 = 0.1$, $\gamma_1 = -0.5$, $N = 1.5$. For $\beta_3\gamma_1 < 0$ and $m < 0$ or $\beta_3\gamma_1 > 0$ and $m > 0$, the solution Eq. (17) is a complex-valued function. The real part of the solution Eq. (17) represents the periodic rogue wave with bell shape (in Figure 10), and the imaginary part represents the periodic rogue wave with singular soliton (in Figure 10). Figure 10 provides the diagram of the solution Eq. (17) for the values $m = 1$, $l = -2$, $\beta_3 = 2$, $\beta_2 = 1$, $\gamma_1 = 0.5$, $N = 1.5$. For $\beta_3\gamma_1 < 0$ and $m < 0$ or $\beta_3\gamma_1 > 0$ and $m > 0$, the solution Eq. (17) is a complex-valued function. Figure 11 illustrates the periodic rogue wave of the solution Eq. (22) for the values $R_2 = -4$, $R_3 = 1$, $m = -3$, $l = 0.5$, $\beta_3 = 2$, $\beta_2 = 1$, $\gamma_1 = 0.5$, $\delta = 0.33$, $N = 1.5$. For $(4\lambda^2 m R_1 R_3 - \lambda^2 m R_2^2 - l^2) > 0$, the solution Eq. (23) is a complex-valued function, and for $(4\lambda^2 m R_1 R_3 - \lambda^2 m R_2^2 - l^2) < 0$, the solution Eq. (23) is a real-valued function. The real part of the solution Eq. (23) represents the kinky

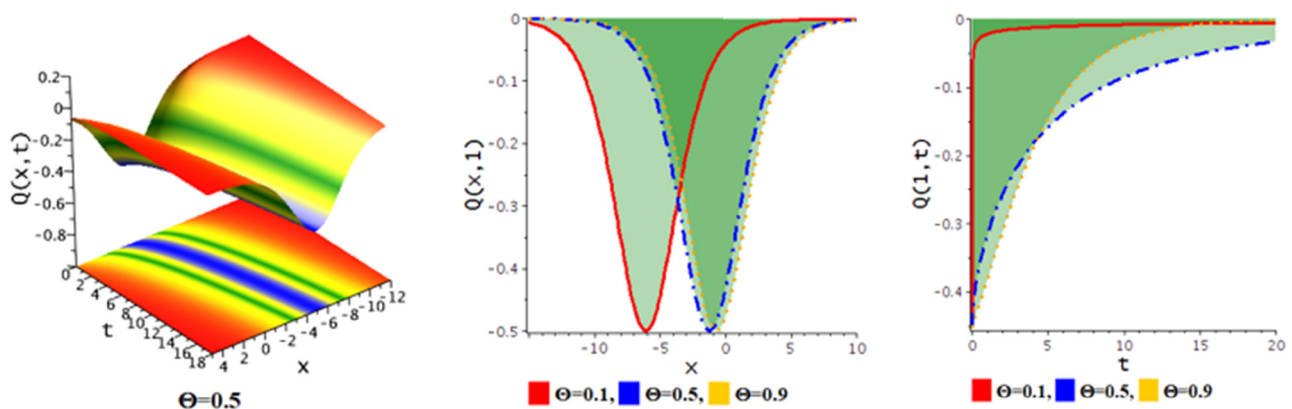


Figure 8: Diagram of dark bell soliton of the solution Eq. (13) for the values $m = -0.5$, $l = 0.5$, $\beta_3 = 1$, $\beta_2 = 0.1$, $\gamma_1 = 0.5$, $N = 1.5$.

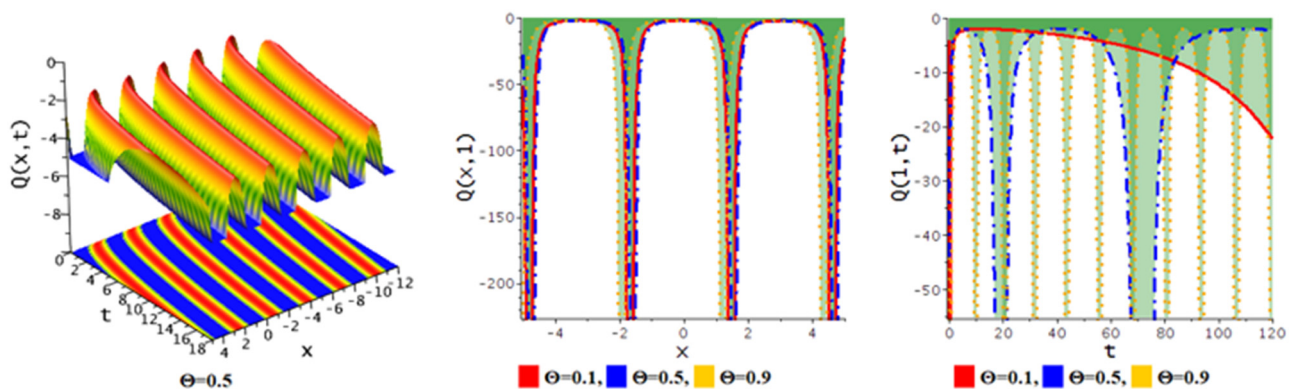


Figure 9: Diagram of bright periodic soliton of the solution Eq. (17) for the values $m = -0.5$, $l = 0.5$, $\beta_3 = -1$, $\beta_2 = 0.1$, $\gamma_1 = -0.5$, $N = 1.5$.

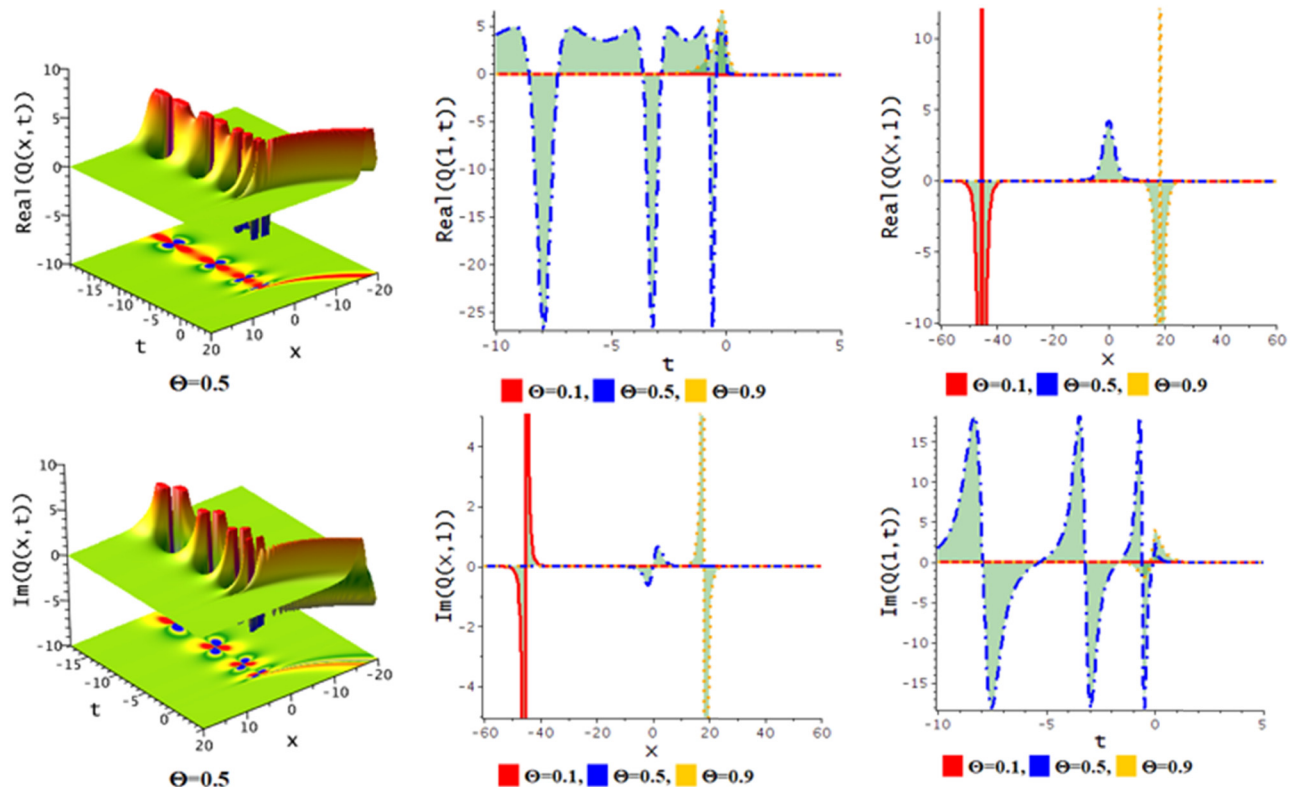


Figure 10: Diagram of the periodic lump wave of the solution Eq. (17) for the values $m = 1$, $l = -2$, $\beta_3 = 2$, $\beta_2 = 1$, $\gamma_1 = 0.5$, $N = 1.5$.

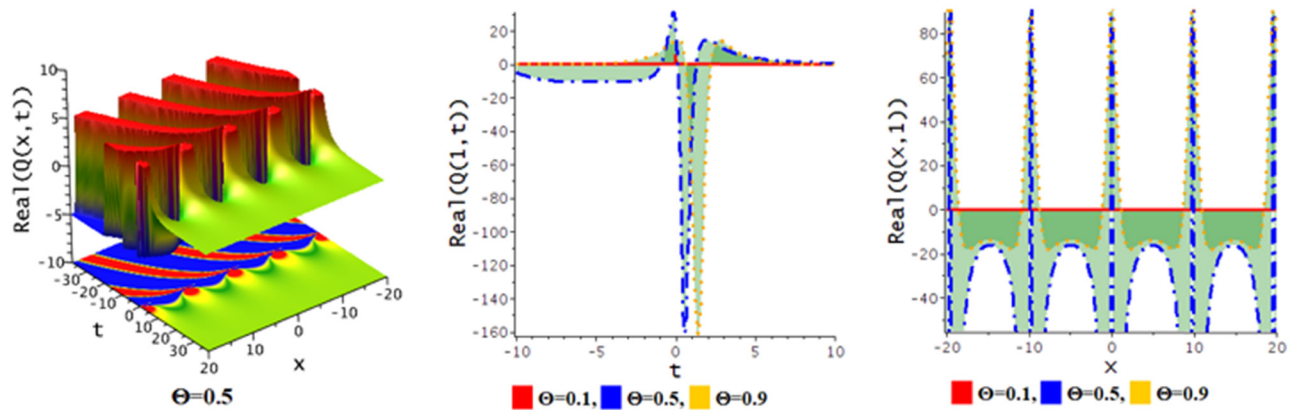


Figure 11: Diagram of the periodic lump wave of solution Eq. (22) for the values $R_2 = -4$, $R_3 = 1$, $m = -3$, $l = 0.5$, $\beta_3 = 2$, $\beta_2 = 1$, $\gamma_1 = 0.5$, $\delta = 0.33$, $N = 1.5$.

periodic rogue wave (in Figure 12), and the imaginary part represents the periodic rogue wave (in Figure 12). Figure 12 provides the diagram of the solution Eq. (23) for the values $\lambda = -0.5$, $m = -3$, $l = 0.05$, $\gamma_0 = 1$, $\gamma_1 = 0.5$, $N = 1.5$. Figure 13 illustrates the periodic wave of the solution Eq. (32) for the values $R_2 = -1$, $R_3 = 1$, $\delta = -0.1$, $\lambda = 0.5$, $m = -1$, $l = 0.33$, $\gamma_0 = 1$, $\gamma_1 = 0.167$, $N = 1.5$.

7 Comparison and novelty

Usman Younas solved the longitudinal wave equation using the modified Sardar sub-equation method [35], showcasing solitons such as bright, dark, singular, bright-dark, bright-singular, complex, and their combinations. Recently, Amit Kumar presented a systematic strategy to

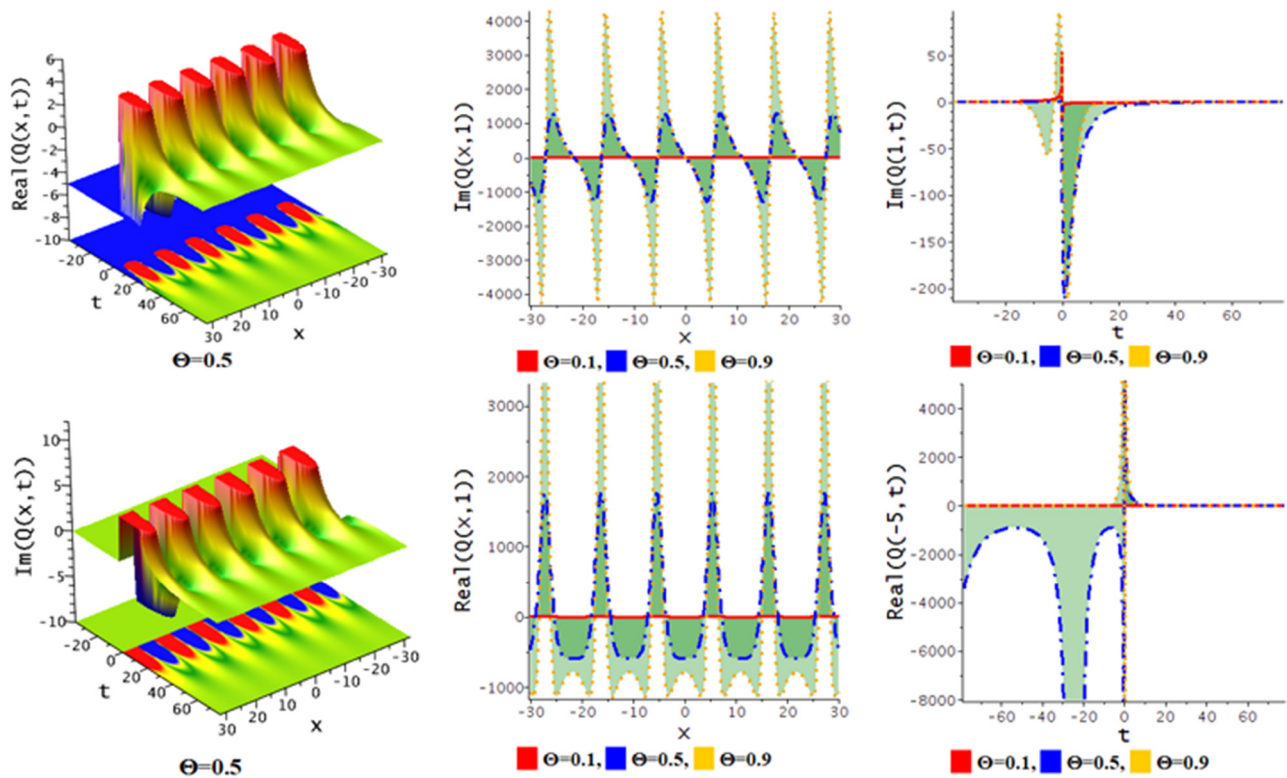


Figure 12: Diagram of the kinky-periodic lump wave of solution Eq. (23) for the values $\lambda = -0.5$, $m = -3$, $l = 0.05$, $\gamma_0 = 1$, $\gamma_1 = 0.5$, $N = 1.5$.

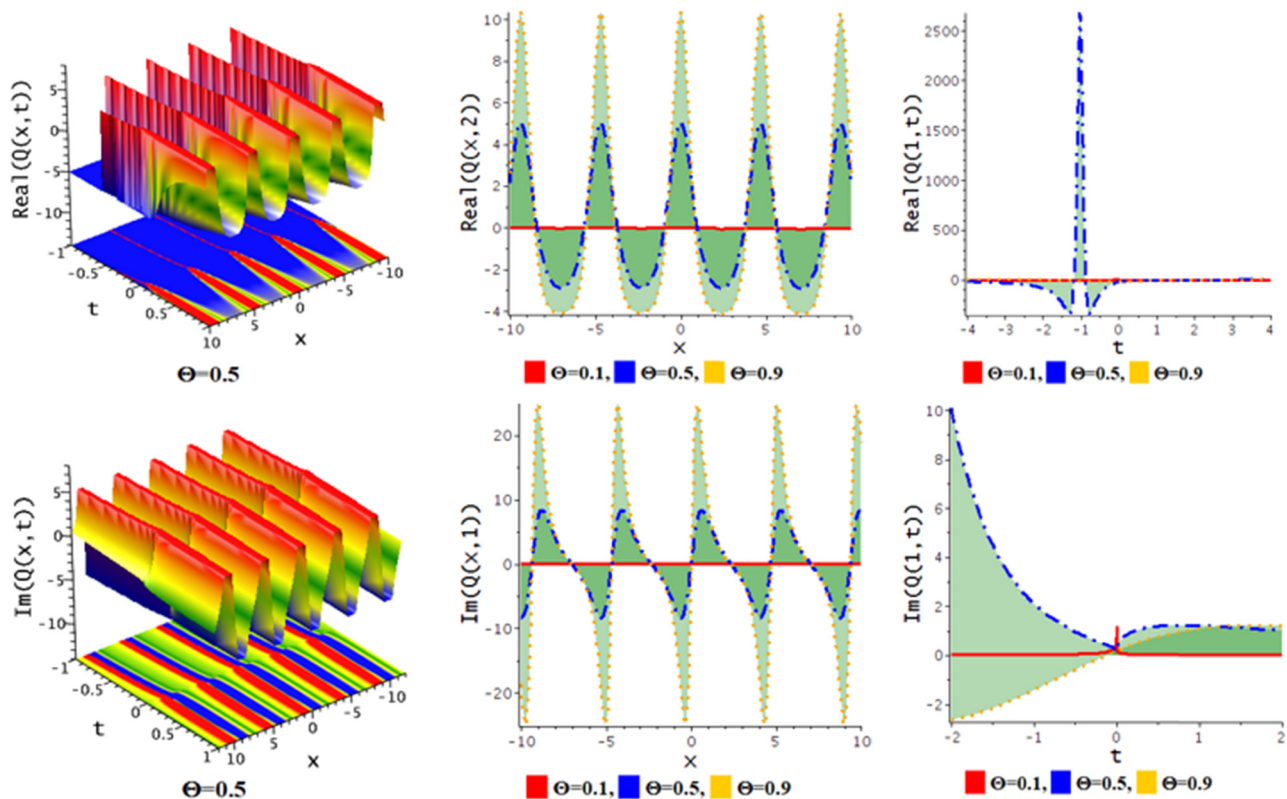


Figure 13: Diagram of the bright periodic of the solution Eq. (32) for the values $R_2 = -1$, $R_3 = 1$, $\delta = -0.1$, $\lambda = 0.5$, $m = -1$, $l = 0.33$, $\gamma_0 = 1$, $\gamma_1 = 0.167$, $N = 1.5$.

uncover the underlying dynamics by deriving solitary wave solutions for LWE using two powerful techniques: the unified method and the UREE method [37]. They provided graphic representations of dynamic wave structures, including traveling waves, mixed periodic waves, solitary solitons, W-shaped waves, kink waves, and interactions between breather waves and kink waves. Onur Alp Ilhan investigated the solitary wave solutions for the (1+1)-dimensional LWE using the sine-Gordon expansion method [39], graphically presenting bell-shaped waves, singular solitons, and periodic wave solutions. Aly R. Seadawy applied the extended trial equation method [40] to exactly solve the LWE, obtaining dark solitons, bright solitons, solitary waves, and periodic solitary wave solutions from the LWE model. In this manuscript, we applied the effective methods: extended modified F-expansion technique to solve the M-fractional LWE equation. For the first time, we expressed the obtained solitary wave solution in terms of logarithmic functions, as well as hyperbolic, trigonometric, and exponential forms. In the numerical discussion section, we presented novel phenomena not found in previous studies. For specific values of the free parameters, we illustrated bright and dark bell-shaped waves, periodic waves, periodic rogue waves with a bell shape, periodic rogue waves with singular soliton, periodic rogue waves, and the interaction of periodic rogue wave and kink shapes. Overall, the applied techniques are simple and effective for finding unique solitary wave solutions, and this manuscript introduces novel phenomena of the M-fractional LWE equation.

8 Conclusion

This work has succeeded in studying the LWE equation with the M-fractional operator. This model describes the propagation of longitudinal waves in a cylindrical rod composed of a material exhibiting elastic, electromagnetic, and magneto-electric. At first, we successfully applied the bifurcation theory and analyze the revealed chaotic behaviors by introducing trigonometric form, Gaussian form, and hyperbolic form perturbations term. We analyze the multi-stability of the proposed model. To explore the chaotic behavior of the LWE model, we illustrate some diagrams such as phase portraits with 3D, 2D, Poincare, and 2D time-series diagrams. Second, we applied an extended modified F-expansion technique to investigate the solitary wave for the LWE equation with the M-fractional operator. The solutions are in the trigonometric, exponential, and hyperbolic functions form. In the numerical discussion section, we presented novel phenomena not found in previous studies. For specific values of the free parameters, we presented

bright and dark bell-shaped waves, periodic waves, periodic rogue waves with a bell shape, periodic rogue waves with singular soliton, periodic rogue waves, and interaction of periodic rogue waves and kink shape. Overall, the applied techniques are simple and effective for finding unique solitary wave solutions, and this manuscript introduces novel phenomena of the M-fLWE equation.

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