Partial Differential Equations in Applied Mathematics 16 (2025) 101323

Contents lists available at ScienceDirect

Partial Differential Equations in Applied Mathematics

journal homepage: www.elsevier.com/locate/padiff

Numerical simulation of multilane traffic flow model based on exponential velocity-density function

Md Rasel Ahmed [a,*], Bitu Joydhar [b], Dr. Md. Laek Sazzad Andallah [c], Prokriti Biswas [c], Jannatul Ferdous [c]

[a] Department of Mathematics, Uttara University, Dhaka, Bangladesh
[b] Department of Mathematical & Physical Sciences, East West University, Dhaka, Bangladesh
[c] Department of Mathematics, Jahangirnagar University, Dhaka, Bangladesh

ARTICLE INFO

Keywords:
Multilane traffic flow model
Explicit upwind scheme
Lax-Friedrichs scheme
Lax-Wendroff scheme
Numerical Solution
Stability Analysis
Rate of Convergence

ABSTRACT

In this paper, we study a multilane traffic flow model incorporating the exponential velocity-density relationship, which yields a nonlinear first-order system of hyperbolic partial differential (PDEs) equations formulated as an initial-boundary value problem (IBVP). This study seeks to construct a physically consistent and computationally efficient model for a multilane traffic flow model, which captures nonlinear vehicular interactions and lane-changing dynamics. Numerical solutions of the multilane traffic flow model, under the specified initial and boundary conditions, are obtained using the finite difference method. The numerical simulations are performed by employing the well-known first-order Explicit Upwind Scheme, the Lax-Friedrichs Scheme, and the second-order Lax-Wendroff Scheme. We evaluate the convergence rate of the numerical solutions, along with detailed well-posedness and stability analysis. To assess the stability and accuracy of three numerical schemes, we compare the velocity and density profiles derived from different schemes. The outcomes of our study have significant applications in traffic management in the real world, including predicting the effect of congestion, optimizing the lane-change effects, and enhancing efficiency through intelligent transportation system intelligent transportation system (ITS) and safety on multilane highways.

1. Introduction

Traffic congestion is a worldwide alarming issue, especially in non-rural areas. It becomes more critical on multilane highways, as interactions between vehicles can lead to significant traffic issues such as delays, accidents, and reduced efficiency. Today, cities are growing and vehicle ownership is increasing, understanding and managing traffic flow on multilane roads becomes essential for effective urban planning and transportation management.

Multilane traffic flow models have emerged as powerful tools to simulate and analyze traffic dynamics under various conditions.[1,2] In this paper, we present a multilane traffic flow model designed to simulate the behavior of vehicles in complex traffic environments. Our model incorporates key factors such as vehicle interactions, lane-changing behavior, velocity variations, and road conditions.[10–14] The foundation of multilane traffic continuum models is a set of conservation laws with source terms, and the derivation of the macroscopic multilane traffic flow model is based on[3-5,15,19]. Describe the traffic flow balance equation and the closure relations on the balance equations to clarify how the multilane traffic model was derived, following[6,8] , etc. Here, we study a multilane traffic flow model for two lanes based on an exponential velocity-density relationship, which yields a non-linear first-order hyperbolic system of partial differential equations as an IBVP.[1] Since finding the analytical solution of these types of PDEs is complicated, we implement numerical solutions of these PDEs by using finite difference schemes, namely the first-order Explicit Upwind Scheme, the Lax-Friedrichs scheme, and the second-order Lax-Wendroff difference scheme. We develop computer programming code for the first-order Explicit Upwind Scheme, the Lax-Friedrichs scheme, and the second-order Lax-Wendroff difference scheme to implement the numerical scheme. We also perform numerical simulations of some qualitative behaviors of traffic flow with respect to various traffic parameters.

Although numerous studies have investigated macroscopic multilane traffic models, most existing approaches are restricted to linear velocity-

* Corresponding author.
E-mail addresses: rasel@uttara.ac.bd (M.R. Ahmed), bitujoydhar@gmail.com (B. Joydhar), andallah@juniv.edu (Dr.Md.L.S. Andallah), prokritimath42@juniv.edu (P. Biswas), jannatulferdous2021ju@gmail.com (J. Ferdous).

https://doi.org/10.1016/j.padiff.2025.101323
Received 16 June 2025; Received in revised form 3 November 2025; Accepted 9 November 2025
Available online 10 November 2025
2666-8181/

Table 1
Nomenclature.

Symbol	Description	Unit
u_k	Vehicle density	Vehicles per km
u_1	Vehicle density of lane 1	Vehicles per km
u_2	Vehicle density of lane 2	Vehicles per km
u_{kmax}	Maximum density or Jam Density	Vehicles per km
q_k	free flow velocity	Vehicle velocity km/second
v_{kmax}	Maximum velocity	km per second
v_{1max}	Maximum velocity of lane 1	km per second
v_{2max}	Maximum velocity of lane 2	km per second
q_k	Traffic Flux	Vehicles per second
q_1	Flux of lane 1	Vehicles per second
q_2	Flux of lane 2	Vehicles per second
T_1^2	Transition rate from lane 1 to lane 2	Vehicles per km per second
T_2^1	Transition rate from lane 2 to lane 1	Vehicles per km per second
t	Time Variable	Measured in seconds
x	space variable	Measured Kilometer

density or Greenshields-type velocity-density relations and seldom incorporate explicit lane-changing dynamics within a coupled framework. To overcome these limitations, the present study introduces a multilane macroscopic traffic flow model based on an exponential velocity-density function that more accurately represents the non-linear characteristics of real traffic. Furthermore, the application of multiple numerical schemes enables a systematic assessment of model stability and accuracy. Thus, this research proposes a physically consistent and computationally efficient framework for analyzing multilane traffic flow, which will reduce the existing gap in the previous research.

The findings of this study offer deeper insights into vehicle interactions and flow patterns on multilane highways. So, the development of more effective traffic management is expected from this study. Table 1

2. Derivation of the governing PDE system for the multilane traffic flow model

The governing equations of the proposed multilane traffic flow model are derived from the principle of vehicle conservation, similar to the conservation of mass in fluid dynamics. The multilane traffic flow model incorporates an exponential velocity-density relation and lane-changing interactions, providing a physically consistent and mathematically justified framework for examining traffic dynamics.

Let $u_1(x,\ t)$ and $u_2(x,\ t)$ represent the traffic densities (vehicles per kilometer) in lane 1 and lane 2, respectively. The relation between velocity and density follows the exponential velocity–density law proposed by Underwood[2]:

$$v_k(\ u_k) = v_{kmax}\left(\exp\left(\frac{-\ u_k}{u_{kmax}}\right)\right),\ k = 1,2$$

For the exponential velocity-density relationship,[2] the flux is

$$q_k(\ u_k) =\ u_k v_{kmax}\left(\exp\left(\frac{-\ u_k}{u_{kmax}}\right)\right),\ k = 1,2$$

Applying the law of conservation of vehicles[1,3,7,17] to each lane yields,

$$\left.\begin{aligned}\frac{\partial u_1}{\partial t} + \frac{\partial q_1}{\partial x} = \frac{u_2}{T_2^1} - \frac{u_1}{T_1^2}\\ \frac{\partial u_2}{\partial t} + \frac{\partial q_2}{\partial x} = \frac{u_1}{T_1^2} - \frac{u_2}{T_2^1}\end{aligned}\right\} \tag{1}$$

3. Numerical solution of the multilane traffic flow model

Due to the complexity of finding the analytical solution, we have to solve the multilane traffic flow model by the finite difference method. The governing PDE (1) is solved numerically using the finite difference method[7,9,13]. We consider our multilane traffic flow model based on an exponential velocity-density relationship with left and right boundary conditions as an initial boundary value problem,

$$\frac{\partial u_1}{\partial t} + \frac{\partial q_1}{\partial x} = \frac{u_2}{T_2^1} - \frac{u_1}{T_1^2};\ a \le x \le b;\ t_o \le t \le T,\ T \ge 0 \tag{3a}$$

$$\frac{\partial u_2}{\partial t} + \frac{\partial q_2}{\partial x} = \frac{u_1}{T_1^2} - \frac{u_2}{T_2^1};\ a \le x \le b;\ t_o \le t \le T,\ T \ge 0 \tag{3b}$$

With Initial Condition:

$$u_1(t_0,x) = (u_1)_0(x);\ a \le x \le b,\ u_2(t_0,x) = (u_2)_0(x);\ a \le x \le b$$

Boundary Condition:

$$u_1(t,a) = (u_1)_a(t);\ t_o \le t \le T,\ u_1(t,b) = (u_1)_b(t);\ t_o \le t \le T$$

$$u_2(t,a) = (u_2)_a(t);\ t_o \le t \le T,\ u_2(t,b) = (u_2)_b(t);\ t_o \le t \le T$$

To obtain a numerical solution, we use three numerical schemes, namely the first-order Explicit Upwind Scheme, the Lax-Friedrichs Scheme, and the second-order Lax-Wendroff Scheme.

4. Explicit upwind scheme

To develop the Explicit Upwind scheme, we approximate the time derivative $\frac{\partial u_1}{\partial t}$ and $\frac{\partial u_2}{\partial t}$ by the forward difference formula as below. According to the Taylor series expansion, we obtain,

$$u_1(t+k,\ x) = u_1(t,\ x) + k\frac{\partial u_1}{\partial t} + \frac{k^2}{2!}\frac{\partial^2 u_1}{\partial t^2} + \ \ldots\ldots\ldots\ldots\ldots$$

$$\Rightarrow \frac{\partial u_1}{\partial t} = \frac{u_1(t+k,x) - u_1(t,x)}{k} - \frac{k}{2!}\frac{\partial^2 u_1}{\partial t^2}\ldots\ldots\ldots\ldots\ldots$$

$$\Rightarrow \frac{\partial u_1}{\partial t} = \frac{u_1(t+k,x) - u_1(t,x)}{k} + O(k)\ldots\ldots\ldots\ldots\ldots\ldots \tag{4.1}$$

Here $O(k)$ is the error introduced by the truncation of the higher-order terms.

Now at (t^n, x_i) assuming, $k = \Delta t,\ t^n + k = t^{n+1}$ & ${u_1}_i^n = u_1(t^n, x_i)$, and for $k \to 0$, then Eqs. (4.1) can be written as

$$\frac{\partial u_1}{\partial t}(t^n, x_i) \approx \frac{u_1(t^n + k, x_i) - u_1(t^n, x_i)}{\Delta t}$$

$$\frac{\partial u_1}{\partial t}(t^n, x_i) \approx \frac{u_1(t^{n+1}, x_i) - u_1(t^n, x_i)}{\Delta t}$$

$$\therefore\ \frac{\partial u_1}{\partial t}(t^n, x_i) \approx \frac{{u_1}_i^{n+1} - {u_1}_i^n}{\Delta t} \tag{4.2}$$

Eqs. (4.2) represents the first-order forward difference approximation of $\frac{\partial u_1}{\partial t}$ at (t^n, x_i).

Similarly, we can write

$$\therefore\ \frac{\partial u_2}{\partial t}(t^n, x_i) \approx \frac{{u_2}_i^{n+1} - {u_2}_i^n}{\Delta t} \tag{4.3}$$

Eqs. (4.3) represents the first-order forward difference approximation of $\frac{\partial u_2}{\partial t}$ at (t^n, x_i).

Again, we discretize the spatial derivative $\frac{\partial q_1}{\partial x}$ and $\frac{\partial q_2}{\partial x}$ by the backward difference formula. Using Taylor's series expansion as,

$$q_1(t,\ x-h) = q_1(t,\ x) - h\frac{\partial q_1}{\partial t} + \frac{h^2}{2!}\frac{\partial^2 q_1}{\partial t^2} - \ \ldots\ldots\ldots\ldots\ldots$$

$$\Rightarrow \frac{\partial q_1}{\partial x} = \frac{q_1(t,x) - q_1(t,x-h)}{h} + \frac{h}{2!}\frac{\partial^2 q_1}{\partial x^2}\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\Rightarrow \frac{\partial q_1}{\partial x} = \frac{q_1(t,x) - q_1(t,x-h)}{h} + O(h)\ldots\ldots\ldots\ldots\ldots\ldots \tag{4.4}$$

Here, $O(h)$ is the error introduced by the truncation of the higher-order terms.

Now at (t^n, x_i) assuming, $h = \Delta x$, $x_i - h = x_{i-1}$ & ${q_1}_i^n = q_1(t^n, x_i)$, and for $h \to 0$, then Eqs. (4.4) can be written as

$$\frac{\partial q_1}{\partial x}(t^n, x_i) \approx \frac{q_1(t^n, x_i) - q_1(t^n, x_i - h)}{\Delta x}$$

$$\frac{\partial q_1}{\partial x}(t^n, x_i) \approx \frac{q_1(t^n, x_i) - q_1(t^n, x_{i-1})}{\Delta x}$$

$$\therefore \frac{\partial q_1}{\partial x}(t^n, x_i) \approx \frac{{q_1}_i^n - {q_1}_{i-1}^n}{\Delta x} \tag{4.5}$$

Similarly, we can write

$$\therefore \frac{\partial q_2}{\partial x}(t^n, x_i) \approx \frac{{q_2}_i^n - {q_2}_{i-1}^n}{\Delta x} \tag{4.6}$$

Eqs. (4.5) and (4.6) represent the first-order backward difference approximation of $\frac{\partial q_1}{\partial x}$ and $\frac{\partial q_2}{\partial x}$ respectively at (t^n, x_i).

Now, substituting Eqs. (4.2) and (4.5) in Eqs. (3a), we obtain

$$\frac{{u_1}_i^{n+1} - {u_1}_i^n}{\Delta t} + \frac{{q_1}_i^n - {q_1}_{i-1}^n}{\Delta x} = \frac{{u_2}_i^n}{T_2^1} - \frac{{u_1}_i^n}{T_1^2}$$

$$\Rightarrow {u_1}_i^{n+1} = {u_1}_i^n - \frac{\Delta t}{\Delta x}\left({q_1}_i^n - {q_1}_{i-1}^n\right) + \Delta t\left(\frac{{u_2}_i^n}{T_2^1} - \frac{{u_1}_i^n}{T_1^2}\right) \tag{4.7}$$

Similarly, substituting Eqs. (4.3) and (4.6) in Eqs. (3b), we obtain

$$\frac{{u_2}_i^{n+1} - {u_2}_i^n}{\Delta t} + \frac{{q_2}_i^n - {q_2}_{i-1}^n}{\Delta x} = \frac{{u_1}_i^n}{T_1^2} - \frac{{u_2}_i^n}{T_2^1}$$

$$\Rightarrow {u_2}_i^{n+1} = {u_2}_i^n - \frac{\Delta t}{\Delta x}\left({q_2}_i^n - {q_2}_{i-1}^n\right) + \Delta t\left(\frac{{u_1}_i^n}{T_1^2} - \frac{{u_2}_i^n}{T_2^1}\right) \tag{4.8}$$

Eqs. (4.5) and (4.6) represent the explicit upwind finite difference schemes applied to the multilane traffic flow model, formulated as the initial-boundary value problem (IBVP) given by Eqs. (3a) and (3b).

For the exponential velocity-density relationship, the traffic flux is

$${q_1}_i^n = q\left({u_1}_i^n\right) = {u_1}_i^n v_{1max}\exp\left(-\frac{{u_1}_i^n}{u_{1max}}\right) \tag{4.9}$$

$${q_2}_i^n = q\left({u_2}_i^n\right) = {u_2}_i^n v_{2max}\exp\left(-\frac{{u_2}_i^n}{u_{2max}}\right) \tag{4.10}$$

$${q_1}_{i-1}^n = q\left({u_1}_{i-1}^n\right) = {u_1}_{i-1}^n v_{1max}\exp\left(-\frac{{u_1}_{i-1}^n}{u_{1max}}\right) \tag{4.11}$$

$${q_2}_{i-1}^n = q\left({u_2}_{i-1}^n\right) = {u_2}_{i-1}^n v_{2max}\exp\left(-\frac{{u_2}_{i-1}^n}{u_{2max}}\right) \tag{4.12}$$

Using Eqs. (4.7) and (4.9) in Eqs. (4.5), we obtain

$$\begin{aligned}{u_1}_i^{n+1} = {u_1}_i^n - v_{1max}\left(\frac{\Delta t}{\Delta x}\right)\left({u_1}_i^n\exp\left(-\frac{{u_1}_i^n}{u_{1max}}\right) - {u_1}_{i-1}^n\exp\left(-\frac{{u_1}_{i-1}^n}{u_{1max}}\right)\right) \\ + \Delta t\left(\frac{{u_2}_i^n}{T_2^1} - \frac{{u_1}_i^n}{T_1^2}\right)\end{aligned} \tag{4.13}$$

Using the equation and (4.10) in equation (4.6), we obtain

$$\begin{aligned}{u_2}_i^{n+1} = {u_2}_i^n - v_{2max}\left(\frac{\Delta t}{\Delta x}\right)\left({u_2}_i^n\exp\left(-\frac{{u_2}_i^n}{u_{2max}}\right) - {u_2}_{i-1}^n\exp\left(-\frac{{u_2}_{i-1}^n}{u_{2max}}\right)\right) \\ + \Delta t\left(\frac{{u_1}_i^n}{T_1^2} - \frac{{u_2}_i^n}{T_2^1}\right)\end{aligned} \tag{4.14}$$

Eqs. (4.13) and (4.14) represent the explicit upwind finite difference scheme of the multilane traffic flow model with the Exponential velocity-density relationship.

5. Lax-Friedrichs scheme

Similarly, to develop the Lax-Friedrichs Scheme, we approximate the time derivative $\frac{\partial u_1}{\partial t}$ and $\frac{\partial u_2}{\partial t}$ by the forward difference formula at any point (t^n, x_i),

$$\frac{\partial u_1}{\partial t}(t^n, x_i) \approx \frac{{u_1}_i^{n+1} - {u_1}_i^n}{\Delta t} \tag{5.1}$$

$$\frac{\partial u_2}{\partial t}(t^n, x_i) \approx \frac{{u_2}_i^{n+1} - {u_2}_i^n}{\Delta t} \tag{5.2}$$

We approximate the spatial derivative $\frac{\partial q_1}{\partial x}$ and $\frac{\partial q_2}{\partial x}$ by the central difference formula. Using Taylor's series expansion as,

$$q_1(t, x+k) = q_1(t,x) + k\frac{\partial q_1}{\partial x} + \frac{k^2}{2!}\frac{\partial^2 q_1}{\partial x^2} + \ \ldots\ldots\ldots\ldots\ldots$$

$$q_1(t, x-k) = q_1(t,x) - k\frac{\partial q_1}{\partial x} + \frac{k^2}{2!}\frac{\partial^2 q_1}{\partial x^2} - \ \ldots\ldots\ldots\ldots\ldots$$

Now, subtracting the above two equations, we obtain

$$q_1(t, x+k) - \ q_1(t, x-k) = \ 2k\frac{\partial q_1}{\partial x} + O\left(k^2\right) \tag{5.3}$$

Here, $O(k^2)$ is the error introduced by the truncation of the higher-order terms.

Now at (t^n, x_i) assuming, $k = \Delta x$, $x_i - k = x_{i-1}$, $x_i + k = x_{i+1}$ & ${q_1}_i^n = q_1(t^n, x_i)$, and for $k \to 0$, then the Eqs. (5.3) can be written as

$$\frac{\partial q_1}{\partial x}(t^n, x_i) \approx \frac{q_1(t^n, x_i + \Delta x) - q_1(t^n, x_i - \Delta x)}{2\Delta x}$$

$$\frac{\partial q_1}{\partial x}(t^n, x_i) \approx \frac{q_1(t^n, x_{i+1}) - q_1(t^n, x_{i-1})}{2\Delta x}$$

$$\therefore \frac{\partial q_1}{\partial x}(t^n, x_i) \approx \frac{{q_1}_{i+1}^n - {q_1}_{i-1}^n}{2\Delta x} \tag{5.4}$$

Similarly, we can write

$$\therefore \frac{\partial q_2}{\partial x}(t^n, x_i) \approx \frac{{q_2}_{i+1}^n - {q_2}_{i-1}^n}{2\Delta x} \tag{5.5}$$

Substituting Eqs. (5.1) and (5.4) into Eqs. (3a) without the source term, we obtain,

$${u_1}_i^{n+1} = {u_1}_i^n - \frac{\Delta t}{2\Delta x}\left({q_1}_{i+1}^n - {q_1}_{i-1}^n\right) \tag{5.6}$$

Substituting Eqs. (5.2) and (5.5) into Eqs. (3a) without the source term, we obtain,

$${u_2}_i^{n+1} = {u_2}_i^n - \frac{\Delta t}{2\Delta x}\left({q_2}_{i+1}^n - {q_2}_{i-1}^n\right) \tag{5.7}$$

Eqs. (5.6) and (5.7) represent the FTCS (Forward Time Central Space) schemes. From Von Neumann stability analysis, it is confirmed that the scheme is unconditionally unstable. But the scheme can be stable for sufficiently small $\frac{\Delta t}{\Delta x}$, if we replace ${u_1}_i^n$ by $\frac{1}{2}\left({u_1}_{i+1}^n + {u_1}_{i-1}^n\right)$ in the Eqs. (5.6), and in that case, the Eqs. (5.6) becomes

$${u_1}_i^{n+1} = \frac{1}{2}\left({u_1}_{i+1}^n + {u_1}_{i-1}^n\right) - \frac{\Delta t}{2\Delta x}\left({q_1}_{i+1}^n - {q_1}_{i-1}^n\right) \tag{5.8}$$

Now, including the source term of (3a) into (5.8), we get

$${u_1}_i^{n+1} = \frac{1}{2}\left({u_1}_{i+1}^n + {u_1}_{i-1}^n\right) - \frac{\Delta t}{2\Delta x}\left({q_1}_{i+1}^n - {q_1}_{i-1}^n\right) + \Delta t\left(\frac{{u_2}_i^n}{T_2^1} - \frac{{u_1}_i^n}{T_1^2}\right) \tag{5.9}$$

Similarly, for Eqs. (3b), we can write,

$$u_{2_i}^{n+1} = \frac{1}{2}\left(u_{2_{i+1}}^n + u_{2_{i-1}}^n\right) - \frac{\Delta t}{2\Delta x}\left(q_{2_{i+1}}^n - q_{2_{i-1}}^n\right) + \Delta t\left(\frac{u_{1_i}^n}{T_1^2} - \frac{u_{2_i}^n}{T_2^1}\right) \tag{5.10}$$

Eqs. (5.9) and (5.10) represent the Lax-Friedrichs finite difference schemes applied to the multilane traffic flow model, formulated as the initial-boundary value problem (IBVP) given by Eqs. (3a) and (3b).

For the exponential velocity-density relationship, we have

$$q_{1_{i+1}}^n = q\left(u_{1_{i+1}}^n\right) = u_{1_{i+1}}^n v_{1max}\exp\left(-\frac{u_{1_{i+1}}^n}{u_{1max}}\right) \tag{5.11}$$

$$q_{2_{i+1}}^n = q\left(u_{2_{i+1}}^n\right) = u_{2_{i+1}}^n v_{2max}\exp\left(-\frac{u_{2_{i+1}}^n}{u_{2max}}\right) \tag{5.12}$$

$$q_{1_{i-1}}^n = q\left(u_{1_{i-1}}^n\right) = u_{1_{i-1}}^n v_{1max}\exp\left(-\frac{u_{1_{i-1}}^n}{u_{1max}}\right) \tag{5.13}$$

$$q_{2_{i-1}}^n = q\left(u_{2_{i-1}}^n\right) = u_{2_{i-1}}^n v_{2max}\exp\left(-\frac{u_{2_{i-1}}^n}{u_{2max}}\right) \tag{5.14}$$

Using Eqs. (5.11) and (5.13) in Eqs. (5.9), we obtain

$$u_{1_i}^{n+1} = \frac{1}{2}\left(u_{1_{i+1}}^n + u_{1_{i-1}}^n\right) - v_{1max}\left(\frac{\Delta t}{2\Delta x}\right)\left(u_{1_{i+1}}^n\exp\left(-\frac{u_{1_{i+1}}^n}{u_{1max}}\right) - u_{1_{i-1}}^n\exp\left(-\frac{u_{1_{i-1}}^n}{u_{1max}}\right)\right) + \Delta t\left(\frac{u_{2_i}^n}{T_2^1} - \frac{u_{1_i}^n}{T_1^2}\right) \tag{5.15}$$

Using Eqs. (5.12) and (5.14) in Eqs. (5.10), we obtain

$$u_{2_i}^{n+1} = \frac{1}{2}\left(u_{2_{i+1}}^n + u_{2_{i-1}}^n\right) - v_{2max}\left(\frac{\Delta t}{2\Delta x}\right)\left(u_{2_{i+1}}^n\exp\left(-\frac{u_{2_{i+1}}^n}{u_{2max}}\right) - u_{2_{i-1}}^n\exp\left(-\frac{u_{2_{i-1}}^n}{u_{2max}}\right)\right) + \Delta t\left(\frac{u_{1_i}^n}{T_1^2} - \frac{u_{2_i}^n}{T_2^1}\right) \tag{5.16}$$

Eqs. (5.15) and (5.16) represent the Lax-Friedrichs finite difference schemes of the multilane traffic flow model with the Exponential velocity-density relationship.

6. Lax-Wendroff scheme

To derive the Lax-Wendroff scheme for the Eqs. (3a) and (3b), we will use the following methods:

- v Lax-Friedrichs scheme at $\left(t^n, x_{i+\frac{1}{2}}\right)$ and $\left(t^n, x_{i-\frac{1}{2}}\right)$ for half step size
- v Leap-frog scheme $\left(t^{n+\frac{1}{2}}, x_i\right)$ for half step size

Step-1: We take the Lax-Friedrichs scheme (5.9) without source term, for half step at $\left(t^n, x_{i+\frac{1}{2}}\right)$ is

$$u_{1_{i+\frac{1}{2}}}^{n+\frac{1}{2}} = \frac{1}{2}\left(u_{1_{i+1}}^n + u_{1_i}^n\right) - \frac{\Delta t}{2\Delta x}\left(q_{1_{i+1}}^n - q_{1_i}^n\right) \tag{6.1}$$

And at $\left(t^n, x_{i-\frac{1}{2}}\right)$ is

$$u_{1_{i-\frac{1}{2}}}^{n+\frac{1}{2}} = \frac{1}{2}\left(u_{1_i}^n + u_{1_{i-1}}^n\right) - \frac{\Delta t}{2\Delta x}\left(q_{1_i}^n - q_{1_{i-1}}^n\right) \tag{6.2}$$

Step-2: (The Leap-frog scheme)

To derive the Leap-frog scheme, we approximate both time and space derivatives by the central difference formula. Using Taylor's series expansion, we have

$$u_1(t+k,x) = u_1(t,x) + k\frac{\partial u_1}{\partial t} + \frac{k^2}{2!}\frac{\partial^2 u_1}{\partial t^2} + \ldots\ldots\ldots\ldots$$

$$u_1(t-k,x) = u_1(t,x) - k\frac{\partial u_1}{\partial t} + \frac{k^2}{2!}\frac{\partial^2 u_1}{\partial t^2} - \ldots\ldots\ldots\ldots\ldots$$

By subtracting the above two equations, we get

$$\frac{\partial u_1}{\partial t} = \frac{u_1(t+k,x) - u_1(t-k,x)}{2k} + O\left(k^2\right) \tag{6.3}$$

Here, $O\left(k^2\right)$ is the error introduced by the truncation of the higher-order terms.

Now at (t^n, x_i) assuming, $k = \Delta t$, $t^n + k = t^{n+1}$, $t^n - k = t^{n-1}$ & $u_{1_i}^n = u_1(t^n, x_i)$, and for $k \to 0$, then the Eqs. (6.3) can be written as

$$\frac{\partial u_1}{\partial t}(t^n, x_i) \approx \frac{u_1(t^n + k, x_i) - u_1(t^n - k, x_i)}{2\Delta t}$$

$$\frac{\partial u_1}{\partial t}(t^n, x_i) \approx \frac{u_1(t^{n+1}, x_i) - u_1(t^{n-1}, x_i)}{2\Delta t}$$

$$\therefore \frac{\partial u_1}{\partial t}(t^n, x_i) \approx \frac{u_{1_i}^{n+1} - u_{1_i}^{n-1}}{2\Delta t} \tag{6.4}$$

Similarly, we approximate the spatial derivative $\frac{\partial q_1}{\partial x}$ by the central difference formula using Taylor's series expansion, as we have

$$\therefore \frac{\partial q_1}{\partial x}(t^n, x_i) \approx \frac{q_{1_{i+1}}^n - q_{1_{i-1}}^n}{2\Delta x} \tag{6.5}$$

Using Eqs. (6.4) and (6.5) in Eqs. (3a) without the source term, we obtain

$$\frac{u_{1_i}^{n+1} - u_{1_i}^{n-1}}{2\Delta t} + \frac{q_{1_{i+1}}^n - q_{1_{i-1}}^n}{2\Delta x} = 0$$

$$\Rightarrow u_{1_i}^{n+1} = u_{1_i}^{n-1} - \left(\frac{\Delta t}{\Delta x}\right)\left(q_{1_{i+1}}^n - q_{1_{i-1}}^n\right)$$

This is known as the Leap-frog scheme.

Now, we take the Leap-frog scheme for half step at $\left(t^{n+\frac{1}{2}}, x_i\right)$

$$u_{1_i}^{n+1} = u_{1_i}^n - \left(\frac{\Delta t}{\Delta x}\right)\left(q_{1_{i+\frac{1}{2}}}^{n+\frac{1}{2}} - q_{1_{i-\frac{1}{2}}}^{n+\frac{1}{2}}\right) \tag{6.7}$$

For the exponential velocity-density relationships, we have $q_k(u) = u_k v_{kmax}\left(\exp\left(\frac{-u_k}{u_{kmax}}\right)\right)$, $k = 1,2;$ which impies

$$q_{1_{i+\frac{1}{2}}}^{n+\frac{1}{2}} = q_1\left(u_{1_{i+\frac{1}{2}}}^{n+\frac{1}{2}}\right) = u_{1_{i+\frac{1}{2}}}^{n+\frac{1}{2}} v_{1max}\exp\left(-\frac{u_{1_{i+\frac{1}{2}}}^{n+\frac{1}{2}}}{u_{1max}}\right) \tag{6.8}$$

$$q_{1_{i-\frac{1}{2}}}^{n+\frac{1}{2}} = q_1\left(u_{1_{i-\frac{1}{2}}}^{n+\frac{1}{2}}\right) = u_{1_{i-\frac{1}{2}}}^{n+\frac{1}{2}} v_{1max}\exp\left(-\frac{u_{1_{i-\frac{1}{2}}}^{n+\frac{1}{2}}}{u_{1max}}\right) \tag{6.9}$$

Now, using Eqs. (6.8) and (6.9) in Eqs. (6.7), we get

$$u_{1_i}^{n+1} = u_{1_i}^{n} - \left(\frac{\Delta t}{\Delta x}\right)\left(u_{1_{i+\frac{1}{2}}}^{n+\frac{1}{2}} v_{1max}\exp\left(-\frac{u_{1_{i+\frac{1}{2}}}^{n+\frac{1}{2}}}{u_{1max}}\right) - u_{1_{i-\frac{1}{2}}}^{n+\frac{1}{2}} v_{1max}\exp\left(-\frac{u_{1_{i-\frac{1}{2}}}^{n+\frac{1}{2}}}{u_{1max}}\right)\right)$$

$$u_{1_i}^{n+1} = u_{1_i}^{n} - v_{1max}\left(\frac{\Delta t}{\Delta x}\right)\left(u_{1_{i+\frac{1}{2}}}^{n+\frac{1}{2}}\exp\left(-\frac{u_{1_{i+\frac{1}{2}}}^{n+\frac{1}{2}}}{u_{1max}}\right) - u_{1_{i-\frac{1}{2}}}^{n+\frac{1}{2}}\exp\left(-\frac{u_{1_{i-\frac{1}{2}}}^{n+\frac{1}{2}}}{u_{1max}}\right)\right) \tag{6.10}$$

Now, using Eqs. (6.1) and (6.2) in Eqs. (6.10), we obtain

$$u_{1_i}^{n+1} = u_{1_i}^{n} - v_{1max}\left(\frac{\Delta t}{\Delta x}\right)\Bigg(\left(\frac{1}{2}\left(u_{1_{i+1}}^{n}+u_{1_i}^{n}\right) - \frac{\Delta t}{2\Delta x}\left(q_{1_{i+1}}^{n}-q_{1_i}^{n}\right)\right)\exp\left(-\frac{\left(\frac{1}{2}\left(u_{1_{i+1}}^{n}+u_{1_i}^{n}\right)-\frac{\Delta t}{2\Delta x}\left(q_{1_{i+1}}^{n}-q_{1_i}^{n}\right)\right)}{u_{1max}}\right) - \left(\frac{1}{2}\left(u_{1_i}^{n}+u_{1_{i-1}}^{n}\right) - \frac{\Delta t}{2\Delta x}\left(q_{1_i}^{n}-q_{1_{i-1}}^{n}\right)\right)\exp\left(-\frac{\left(\frac{1}{2}\left(u_{1_i}^{n}+u_{1_{i-1}}^{n}\right)-\frac{\Delta t}{2\Delta x}\left(q_{1_i}^{n}-q_{1_{i-1}}^{n}\right)\right)}{u_{1max}}\right)\Bigg) + \Delta t\left(\frac{u_{2_i}^{n}}{T_2^1} - \frac{u_{1_i}^{n}}{T_1^2}\right) \tag{6.11}$$

Similarly, for the Eqs. (3b), we obtain

$$u_{2_i}^{n+1} = u_{2_i}^{n} - v_{2max}\left(\frac{\Delta t}{\Delta x}\right)\Bigg(\left(\frac{1}{2}\left(u_{2_{i+1}}^{n}+u_{2_i}^{n}\right) - \frac{\Delta t}{2\Delta x}\left(q_{2_{i+1}}^{n}-q_{2_i}^{n}\right)\right)\exp\left(-\frac{\left(\frac{1}{2}\left(u_{2_{i+1}}^{n}+u_{2_i}^{n}\right)-\frac{\Delta t}{2\Delta x}\left(q_{2_{i+1}}^{n}-q_{2_i}^{n}\right)\right)}{u_{2max}}\right) - \left(\frac{1}{2}\left(u_{2_i}^{n}+u_{2_{i-1}}^{n}\right) - \frac{\Delta t}{2\Delta x}\left(q_{2_i}^{n}-q_{2_{i-1}}^{n}\right)\right)\exp\left(-\frac{\left(\frac{1}{2}\left(u_{2_i}^{n}+u_{2_{i-1}}^{n}\right)-\frac{\Delta t}{2\Delta x}\left(q_{2_i}^{n}-q_{2_{i-1}}^{n}\right)\right)}{u_{2max}}\right)\Bigg) + \Delta t\left(\frac{u_{2_i}^{n}}{T_2^1} - \frac{u_{1_i}^{n}}{T_1^2}\right) \tag{6.12}$$

Where $q_{1_i}^{n} = q_1\left(u_{1_i}^{n}\right) = u_{1_i}^{n} v_{1max}\exp\left(-\frac{u_{1_i}^{n}}{u_{1max}}\right)$

$$q_{2_i}^{n} = q_2\left(u_{2_i}^{n}\right) = u_{2_i}^{n} v_{2max}\exp\left(-\frac{u_{2_i}^{n}}{u_{2max}}\right)$$

$$q_{1_{i\pm1}}^{n} = q_1\left(u_{1_{i\pm1}}^{n}\right) = u_{1_{i\pm1}}^{n} v_{1max}\exp\left(-\frac{u_{1_{i\pm1}}^{n}}{u_{1max}}\right)$$

$$q_{2_{i\pm1}}^{n} = q_2\left(u_{2_{i\pm1}}^{n}\right) = u_{2_{i\pm1}}^{n} v_{2max}\exp\left(-\frac{u_{2_{i\pm1}}^{n}}{u_{2max}}\right)$$

Eqs. (6.11) and (6.12) represent the Lax-Wendroff scheme of the multilane traffic flow model based on the exponential velocity-density relationship.

7. Well-Posedness and stability condition of numerical schemes

The implementation of the Explicit Upwind Scheme, the Lax-Friedrichs scheme, and the Lax-Wendroff scheme is not straightforward, because the multilane traffic flow model, we are considering that the vehicles are moving in only one direction[8,16,18], so the characteristic speed $c_k\left(u_{k_i}^n\right) = q_k'\left(u_{k_i}^n\right)$ must be non-negative, which implies

$$q_k'\left(u_{k_i}^n\right) \geq 0$$

In particular, if we consider the exponential velocity-density relation, then we can write

$$q_k\left(u_{k_i}^n\right) = u_{k_i}^n v_{kmax}\left(\exp\left(-\frac{u_{k_i}^n}{u_{kmax}}\right)\right),\ k = 1,\ 2$$

$$\Rightarrow q_k'\left(u_{k_i}^n\right) = v_{kmax}\left(\exp\left(-\frac{u_{k_i}^n}{u_{kmax}}\right)\right)\left(1-\frac{u_{k_i}^n}{u_{kmax}}\right) \geq 0,\ k = 1,\ 2 \tag{7.1}$$

Since $v_{kmax} > 0$ and $\exp\left(-\frac{u_{k_i}^n}{u_{kmax}}\right) > 0$, so the above inequality (7.1) implies that,

$$\Rightarrow \left(1-\frac{u_{k_i}^n}{u_{kmax}}\right) \geq 0,\ k = 1,\ 2$$

$$\Rightarrow u_{kmax} \geq u_{k_i}^n,\ k = 1,\ 2$$

This is the required well-posed condition.

Now, from Eqs. (7.1), noting that $0 \leq u_{k_i}^n \leq u_{kmax}$, we have $0 \leq q_k'\left(u_{k_i}^n\right) \leq v_{kmax}$

It follows that the characteristic speed $c_k\left(u_{k_i}^n\right) = q_k'\left(u_{k_i}^n\right)$ is bounded and satisfies the well-posedness condition.

CFL Stability Condition of the Explicit Upwind Scheme:

The CFL stability condition of the explicit upwind scheme is

$$\max_i \left|q_k'\left(u_{k_i}^n\right)\right|\left(\frac{\Delta t}{\Delta x}\right) \leq 1$$

With the exponential flux $\max_i \left|q_k'\left(u_{k_i}^n\right)\right| = v_{kmax}$, so the above inequality gives the practical condition $\frac{v_{kmax}\Delta t}{\Delta x} \leq 1$, where $k = 1,\ 2$. This is the well-posedness and the CFL stability condition of the explicit upwind scheme for both lanes.

CFL Stability Condition of the Lax-Friedrichs Scheme:

The CFL stability condition of the Lax-Friedrichs scheme is

$$\max_i \left|q_k'\left(u_{k_i}^n\right)\right|\left(\frac{\Delta t}{\Delta x}\right) \leq 1$$

With the exponential flux $\max_i \left|q_k'\left(u_{k_i}^n\right)\right| = v_{kmax}$, so the above inequality gives the practical condition $\frac{v_{kmax}\Delta t}{\Delta x} \leq 1$, where $k = 1,\ 2$.

CFL Stability Condition of the Lax-Wendroff Scheme:

The CFL stability condition of the Lax-Wendroff scheme is

$$\max_i \left|q_k'\left(u_{k_i}^n\right)\right|\left(\frac{\Delta t}{\Delta x}\right) \leq 1$$

With the exponential flux $\max_i \left|q_k'\left(u_{k_i}^n\right)\right| = v_{kmax}$, so the above inequality gives the practical condition $\frac{v_{kmax}\Delta t}{\Delta x} \leq 1$, where $k = 1,\ 2$.

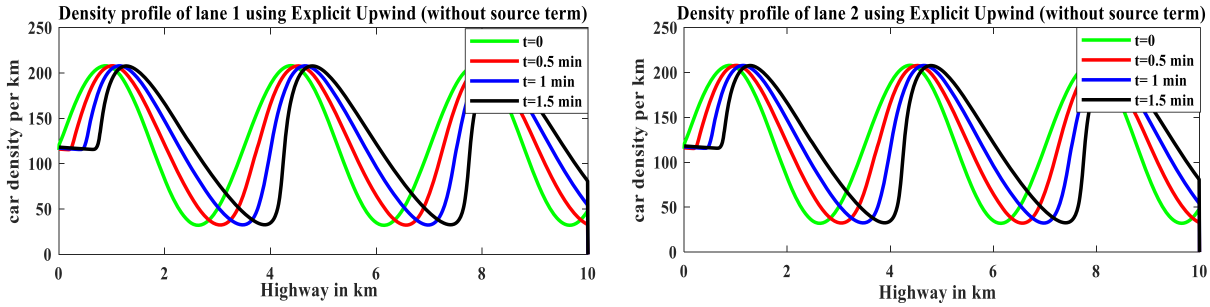

Fig. 1. Density profile of Lane 1 and Lane 2 without including transition rate / source term.

It is mentioned that if the lane-changing rates are large (small T), a Courant safety factor

$\alpha \leq 1$ is applied,

$$\Delta t \leq min\left(\frac{\Delta x}{v_{1max}}, \frac{\Delta x}{v_{1max}}, \alpha T_{min}\right), \; T_{min} = min\left(T_1^2, \; T_2^1\right)$$

This ensures the stability of the coupled multilane system with the explicit source terms.

8. Numerical simulation and result discussion

The density and velocity profiles are obtained using the first-order explicit upwind scheme, the Lax-Friedrichs scheme, and the Lax-Wendroff scheme. To incorporate these schemes, we consider the following assumptions:

- ➢ The highway is assumed to have two lanes with a total length of 10 km, where our spatial domain is [0, 10] km.
- ➢ We perform a numerical experiment for 1.5 min.
- ➢ Periodic initial and boundary conditions were applied.

To implement the explicit upwind difference scheme, the Lax-Friedrichs scheme, and the Lax-Wendroff difference schemes for the exponential velocity density relationship initial and boundary conditions are considered as below:

Initial condition : $u_0(x) = A * \sin(a * x(i)) + B$

Left boundary Condition : $u_a(t) = C * \sin(b * t) + D$

Where $u_0(x)$ means initial density, *i.e*, density at time $t = 0$, and $u_a(t)$ stands for density at the boundary, *i.e.* at the point $x = a$.

- ➢ At the right boundary, we consider the Neumann boundary condition.
- ➢ The exponential velocity–density function introduced by the Underwood model is assumed.
- ➢ We perform a numerical experiment for 1.5 minutes; we consider 1500 spatial grid points with a spatial step size $\Delta x = 0.0067$ km and 360 temporal grid points with a step size $\Delta t = 0.25$ second
- ➢ For the multilane traffic flow model, we consider the car transition rate from lane 1 to lane 2 is 30% and from lane 2 to lane 1 is 45%.
- ➢ We have considered the maximum velocity to be $v_{1max} = v_{2max} = 60km/hour = 0.0167km/sec$. It is noted that, to satisfy the CFL condition, the velocity is expressed in units of kilometers per second.

Also, maximum density u_{kmax} is estimated by the following equations

$$u_{kmax} = k * \max(u(0, x)); \; k \geq 1$$

We have chosen k=8; which implies $u_{kmax} = u_{1max} = u_{2max} = 383$ vehicles per km in the spatial domain [0, 10], whereas maximum velocity and maximum density are considered as constant in the whole 10 km chosen highway.

In this section, we have presented the density profiles of the multilane traffic flow model (two-lane) without source term/ transition rate, with source term/ transition rate, and also presented velocity profiles by using the first-order explicit upwind scheme, the Lax-Friedrichs scheme, and the Lax-Wendroff scheme. The convergence of schemes is also shown in this section.

In Fig. 1–12, we present numerical experiments of density profile without including transition rate, density profile with including transition rate, velocity profile and convergence for lane 1 and lane 2 for 1.5 min or 90 s of multilane traffic flow model to visualize the behavior of density, velocity and convergence with respect to time and space by using Explicit Upwind scheme, Lax-Friedrichs scheme and Lax-Wendroff scheme respectively. After including the transition rate, we observe that the density profile of lane 1 is increasing, whereas the density profile of lane 2 is decreasing, and we also observe that the velocity profile of lane 2 is higher compared to the velocity profile of lane 1 because the difference in the car density of lane 1 is higher than the density of lane 2.

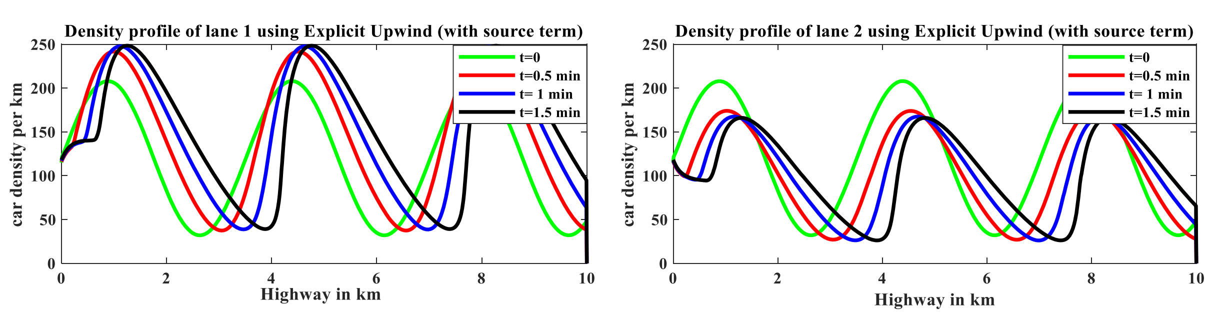

Fig. 2. Density profile of Lane 1 and Lane 2 with including transition rate / source term.

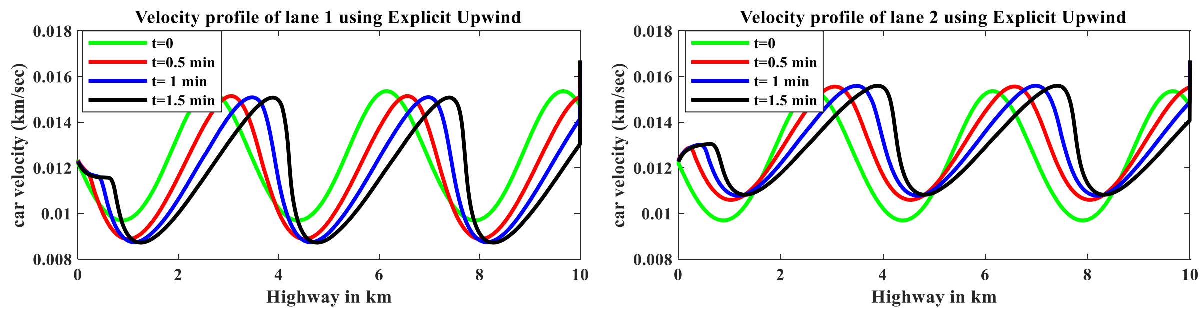

Fig. 3. Velocity profile of Lane 1 and Lane 2 with including transition rate/ source term.

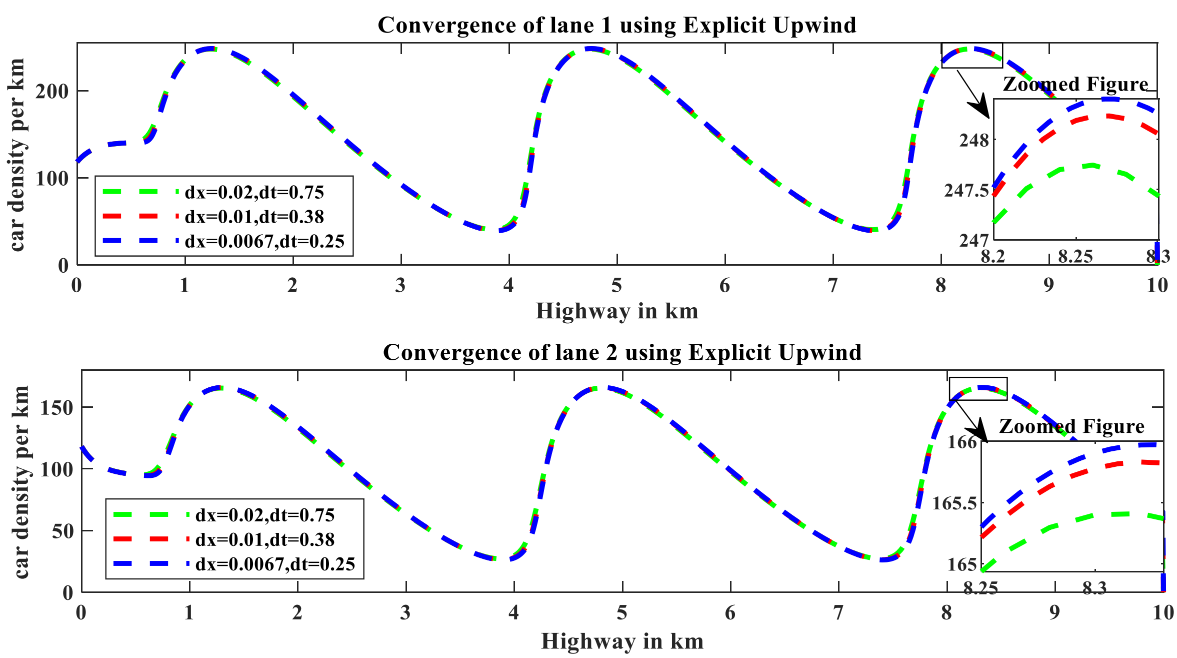

Fig. 4. Convergence of Lane 1 and Lane 2 including transition rate/ source term.

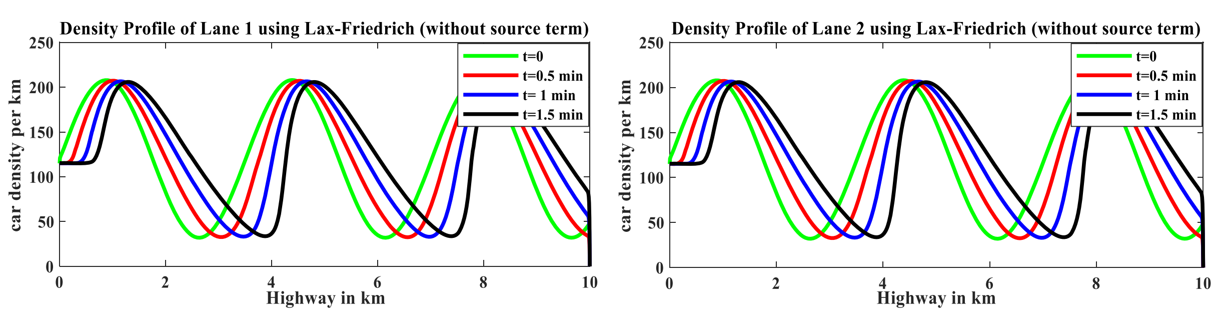

Fig. 5. Density profile of Lane 1 and Lane 2 without including transition rate/ source term.

9. Comparison among Explicit Upwind, Lax-Friedrichs, and Lax-Wendroff schemes

In this section, we would like to inquire which scheme among the explicit upwind scheme, the Lax-Friedrichs, and the Lax-Wendroff for the multilane traffic flow model provides us with better results. In this part, the schemes for various traffic flow parameters of the multilane traffic flow model are compared and examined.

Based on the assumption stated in Section 8, the numerical results of the density profile (Fig. 13) and velocity profile (Fig. 14) for the explicit upwind scheme, the Lax-Friedrichs scheme, and the Lax-Wendroff scheme in the multilane traffic flow models are provided.

In Fig. 13, we have presented a comparison of density among the explicit upwind scheme, the Lax-Friedrichs scheme, and the Lax-Wendroff scheme of lane 1 and lane 2, respectively, whereas in Fig. 14, we have presented a comparison of velocity among the explicit upwind, the Lax-Friedrichs, and the Lax-Wendroff schemes of lane 1 and lane 2, respectively.

From Figs. 13 and 14, by comparing the density profiles and velocity profiles, we find that the Lax-Wendroff scheme gives a better approximation compared to the Explicit Upwind scheme and Lax-Friedrichs scheme for the multilane traffic flow model.

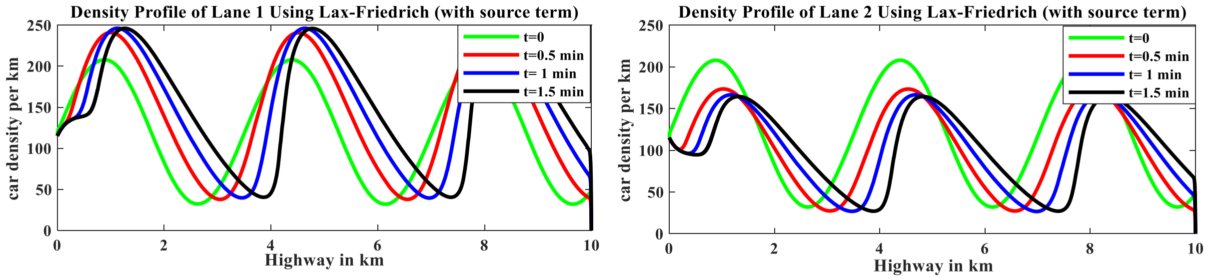

Fig. 6. Density profile of Lane 1 and Lane 2 with including transition rate/ source term.

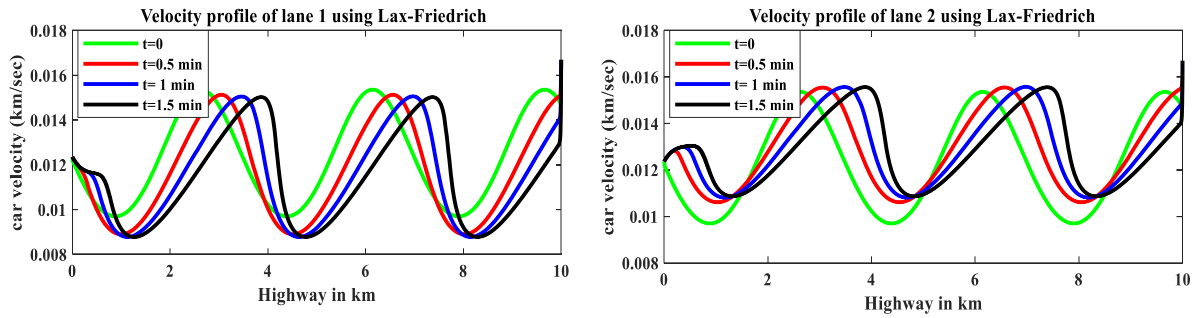

Fig. 7. Velocity profile of Lane 1 and Lane 2 with including transition rate/ source term.

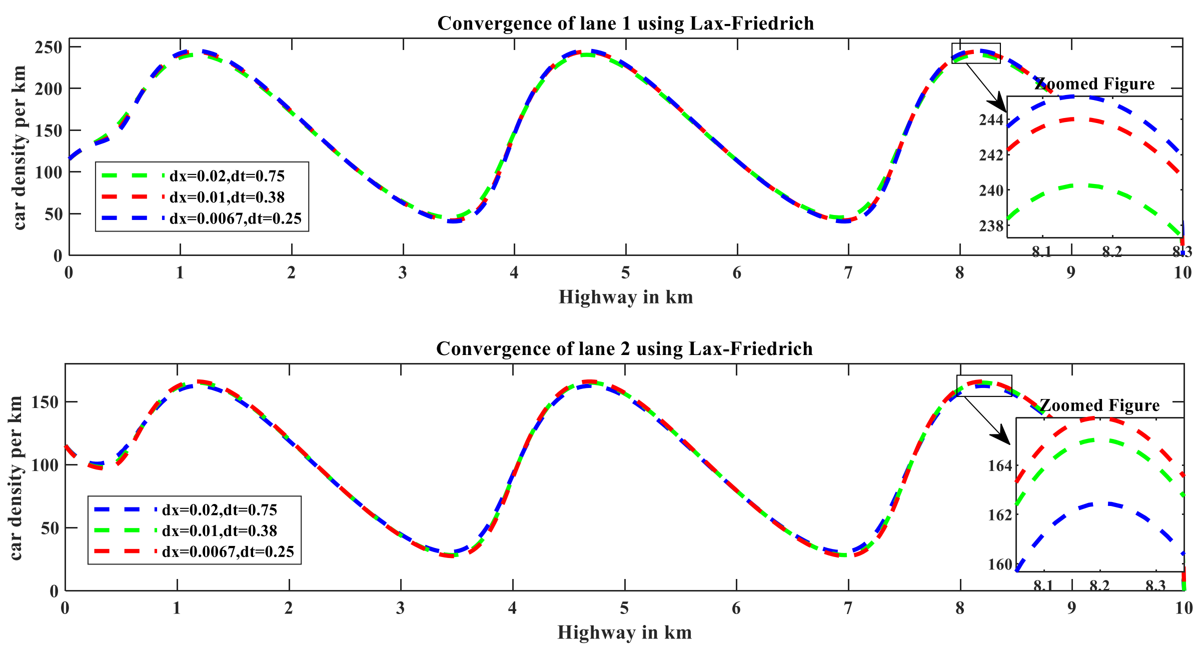

Fig. 8. Convergence of Lane 1 and Lane 2 including transition rate/ source term.

10. Conclusion

- Three numerical schemes, namely the Explicit Upwind scheme, the Lax-Friedrichs scheme, and the Lax-Wendroff scheme, were applied to simulate the multilanetraffic flow model.
- All schemes produced stable and consistent solutions for traffic density and velocity.
- Convergence analysis confirmed that the numerical solutions converge smoothly with grid refinement, validating the stability and reliability of the Explicit Upwind scheme, the Lax-Friedrichs scheme, and the Lax-Wendroff scheme.
- The Lax-Wendroff scheme is stable and demonstrates superior accuracy and faster convergence compared to the other schemes.
- The Lax-Friedrichs scheme is more diffusive but computationally efficient, making it suitable for large-scale or real-time traffic simulations.
- The Explicit Upwind scheme is stable and provides medium accuracy compared to the Lax-Friedrichs scheme and the Lax-Wendroff scheme.
- Finally, we may conclude that the Lax-Wendroff scheme is recommended for high accuracy compared to the Explicit upwind and Lax-Friedrichs schemes, whereas the Lax-Friedrichs scheme provides a

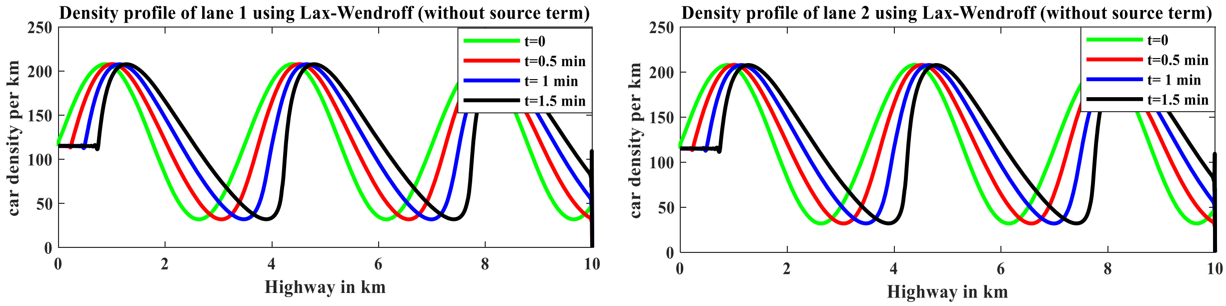

Fig. 9. Density profile of Lane 1 and Lane 2 without including transition rate/ source term.

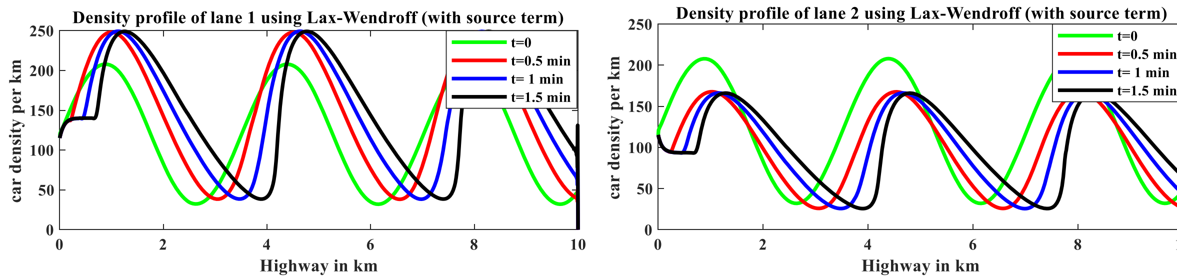

Fig. 10. Density profile of Lane 1 and Lane 2 with including transition rate/ source term.

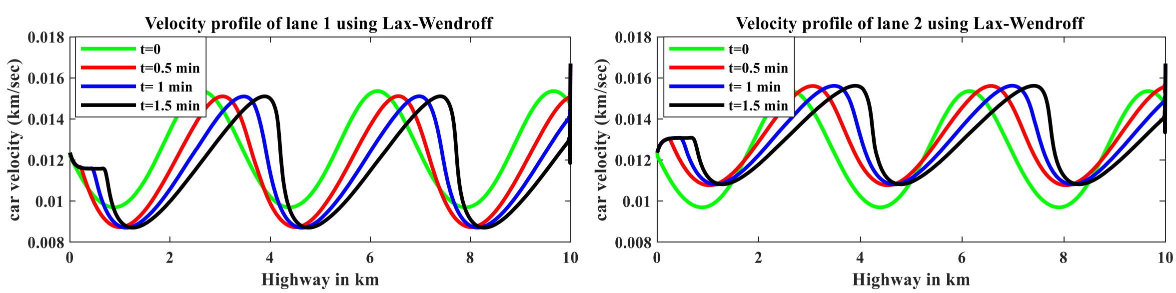

Fig. 11. Velocity profile of Lane 1 and Lane 2 with including transition rate/ source term.

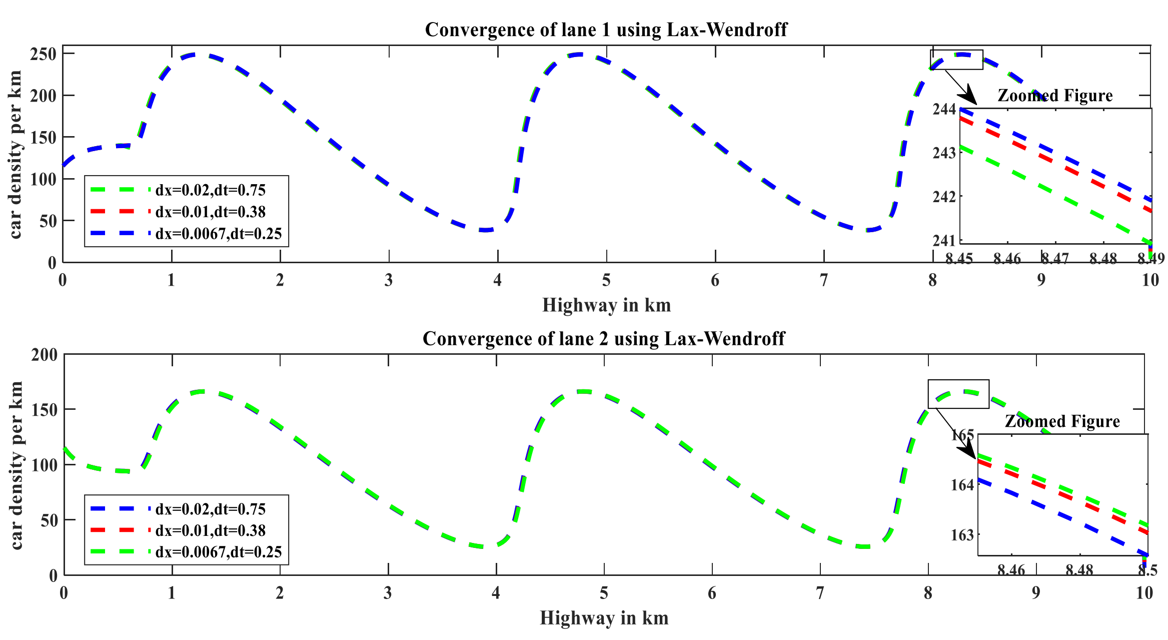

Fig. 12. Convergence of Lane 1 and Lane 2 including transition rate.

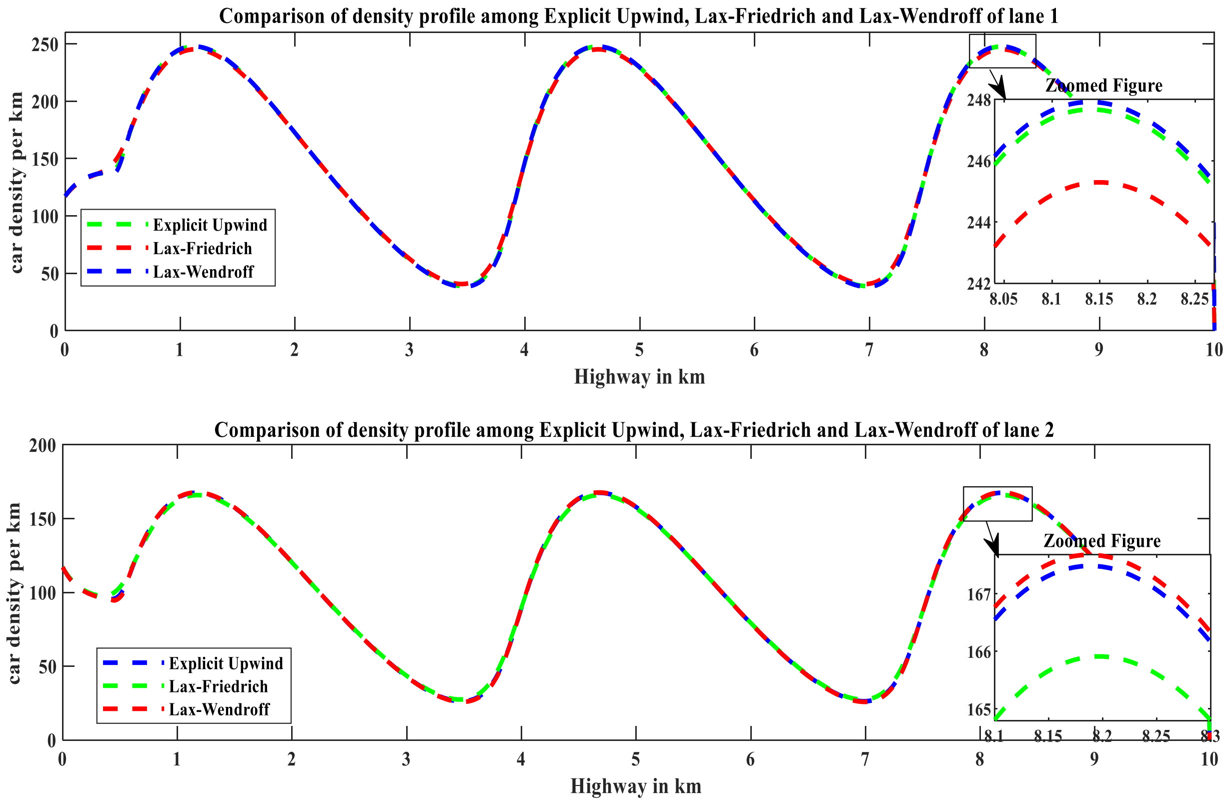

Fig. 13. Comparison of density profile among Explicit Upwind, Lax-Friedrichs and Lax-Wendroff schemes.

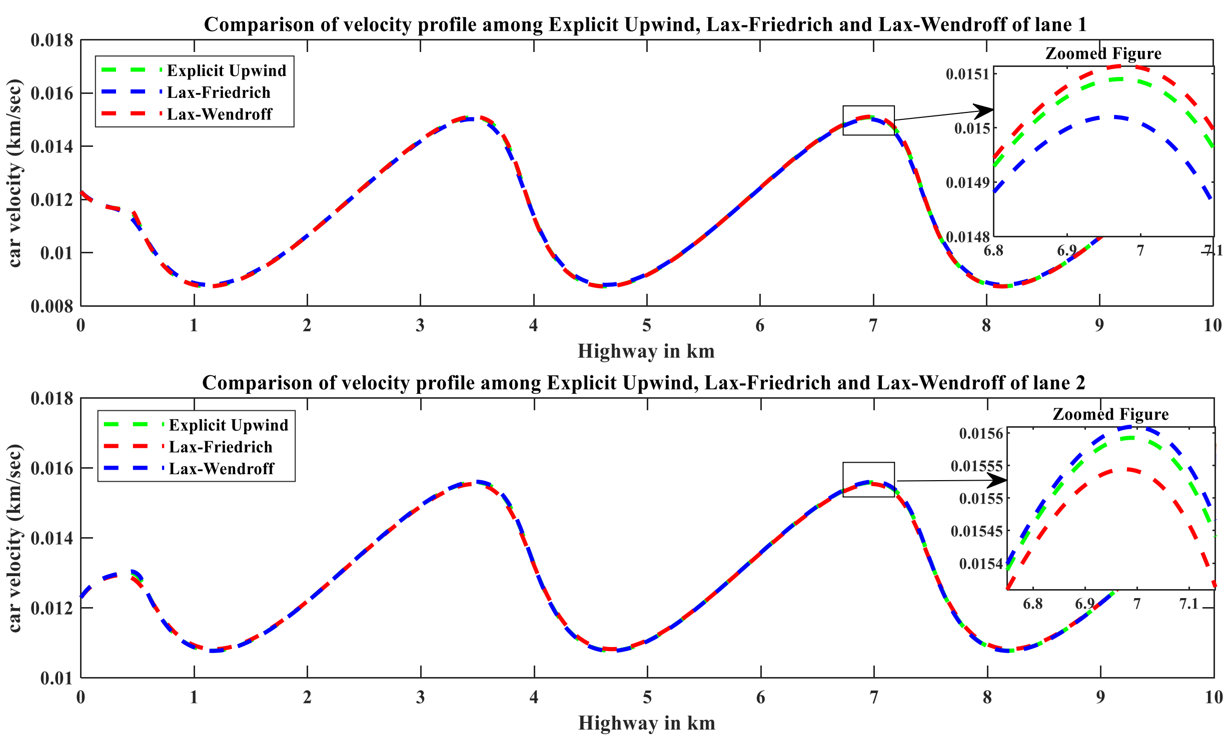

Fig. 14. Comparison of velocity profile among Explicit Upwind, Lax-Friedrichs and Lax-Wendroff schemes.

practical balance between accuracy and efficiency compared to the Explicit upwind and Lax-Wendroff schemes

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

References

1. Daganzo CF. A behavioral theory of MultiLane traffic flow. Part I: long homogeneous freeway sections. *Transp Res B*. 2002;36(2):131–158.
2. Haque Mojammel, Sultana Mariam, Andallah LS. Numerical simulation of diffusion type traffic flow model using second-order lax-wendroff scheme based on exponential velocity density function. *Am J Comput Math*. 2023;13:398–411.
3. Herty M, Moutari S, Visconti G. Macroscopic modeling of multilane motorways using a TwoDimensional SecondOrder model of traffic flow. *SIAM J Appl Math*. 2018;78(4):2252–2278.
4. Greenshields BD. A study of traffic capacity. *Highw Res Board*. 1934;14:448–477.
5. Klar A, Kuhene RD, Wegener R. A hierarchy of models for multilane vehicular traffic I. Modeling. *SIAM Appl Math*. 1999;59:983–1001.
6. Lighthill MJ, Whitham GB. On kinematic waves I. Flood movement in long rivers. *Proc R Soc Lond A Math Phys Sci*. 1955;229:281–316.
7. Ali MdShajib, Andallah LS. Numerical solution of a multilane fluid dynamic traffic flow model with three lanes. *IOSR J Math*. 2019;15(2 Ser. III):01–10. Mar – Apr.
8. Corli RM, Colombo. Well posed-ness of multilane traffic models. *Ann Univ Ferrara Sez VII SciMat*. 2006;52:291–301.
9. Hasan Mahmudul, Sultana S, Andallah LS, Azam T. Lax-Friedrich scheme for the numerical simulation of a traffic flow model based on a nonlinear velocity density relation. *Am J Comput Math*. 2015;5:186–194.
10. Holden H, Risebro NH. Models for dense multilane vehicular traffic. *SIAM*. 2019;51: 3694–3713.
11. Kabir MH, Andallah LS. Numerical solution of a multilane traffic flow model", GANIT. *J Bangladesh Math Soc*. 2013;33 (ISSN 1606-3694).
12. Bretti G, Natalini R, Piccoli B. *A Fluid-Dynamic Traffic Model on Road Networks*. Barcelona, Spain: Compute Method Eng; 2007.
13. Andallah LS, Ali MS, Akhter J, Gani MO, Pandit MK. A finite difference scheme of traffic flow model based on linear velocity-density function. *Jahangirnagar Univ J Sci*. 2008;32(1):57–67.
14. Davis LC. Multilane simulations of traffic phases. *Phys Rev E*. 2004;69, 016108.
15. Kabir MH, Gani MO, Andallah LS. Numerical simulation of a mathematical traffic flow model based on a non-linear velocity density function. *J Bangladesh Acad Sci*. 2010;34(1):15–22.
16. Azad TMAK, Andallah LS. Stability analysis of finite difference scheme for an advection-diffusion equation. *Bangladesh J Sci Res*. 2016;29(2):143–151.
17. Vladimir S, Dirk H. Macroscopic Dynamics of Multi-Lane Traffic. *Phys Rev E*. 1999; 59:6328–6339.
18. Azam T, Pandit MK, Andallah LS. NUMERICAL SIMULATION OF A SECOND ORDER TRAFFIC FLOW MODEL. *GANIT J Bangladesh Math Soc*. 2014;34:101–111 (ISSN 1606-3694).
19. Colombo RM, Corli A, Communications to SIMAI Congress. On Multilane Traffic Flow. 2006;1. https://doi.org/10.1685/CSC06056.