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Diverse analytical wave solutions of plasma physics and water wave equations

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A R T I C L E I N F O

Keywords:
Advanced auxiliary equation method
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Wave solutions
Soliton solutions

A B S T R A C T

In plasma physics and water waves, the Gilson-Pickering equation is an important unidirectional wave propagation model. A large number of analytic wave solutions have been established in the form of hyperbolic, trigonometric, exponential, and rational functions using the advanced auxiliary equation approach in this study. The solutions obtained have been compared with the solutions available in the literature, and it is observed that we have established further wave solutions than the solutions determined by the other methods, namely the (G'/G^2)-expansion method, sinh-Gordon approach, etc. This method uses a homogeneous balance rule to estimate a polynomial-type solution and delivers an order. A traveling wave transformation has been used to convert the governing equation into a nonlinear differential equation. Each solution includes a variety of parameters related to the model and method. Using different parameter values, the nature of wave solutions is defined by three-dimensional (3D) and two-dimensional (2D) wave profiles. The solitons change their nature and location for the particular values of the dispersion coefficient α and free parameter κ, and shown in 2D figures to illustrate the different forms of solitons.

Introduction

We have discussed the background information and literature review in sub-section 1.1. A description of the model is provided in sub-section 1.2. The organization of the article is covered in Sub-section 1.3.

Background and literature review

Nonlinear partial differential equations (NLPDEs) are one of the most important topics in research, and they are considered an important illustration for understanding nonlinear tangible phenomena. Therefore, the analysis of nonlinear phenomena is one of the most exciting topics in the current era of research. In ocean engineering [1], fluid mechanics [2,3], plasma physics [4], particle physics [5], biological physics [6], optical fibers [7], and many other scientific fields, the NLPDEs have emerged. The NLPDEs play an essential role in determining the physical behaviors and dynamic processes of real events.

It has recently become one of the most intriguing issues for the investigators, and they have focused on finding analytical wave solutions to the NLPDEs. These solutions play a significant role in science and engineering and can be utilized to study the nature of various models. As a result, several effective and efficient methods have been established to extract analytic solutions for nonlinear physical models, such as the Hirota bilinear formulation [8], the Darboux transformation approach [9], the sub-equation method [10], the modified simple equation scheme [11], the (w/g)-expansion technique [12], the enhanced Kudryashov method [13,14], the $\exp(-\varphi(\xi))$-expansion method [15,16], the Jacobi elliptic function method [17], the extended modified direct algebraic method [18,19], the Lie symmetry analysis method [20,21], the enhanced (G'/G)-expansion method [22,23], the auxiliary equation method [24], the modified extended exponential rational function approach [25,26], the enhance MSE scheme [27], the sine-Gordon approach [28], and other schemes.

In 1995, Gilson and Pickering (GP) were first introduced this equation in [29], and factorization and Painleve analysis were investigated. Clarkson et al., [30] explored the symmetry reduction of the GP equation two years later. In 2009, Chen et al., [31] used the bifurcation method to examine the qualitative behavior and exact solutions to the

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Gilson-Pickering equation. Ebadi et al., [32] explored the conservation laws of the GP equation by using the (G'/G) method to determine soliton solutions. Recently, several scholars have examined the exact traveling wave solutions to the GP model by using different schemes, such as the simplified (G'/G)-expansion method [33], the first integral method [34], the Bernoulli sub-equation function method [35], the sinh-Gordon function and $(G'/G,\ 1/G)$-expansion schemes [36], the generalized exponential rational function method [37], the sine-Gordon expansion method [38], More recently, Bilal et al., [39] attained the exact wave solutions to the GP equation by (G'/G^2)-expansion and expansion function methods; and Rani et al., [40] investigated the soliton wave structure of the GP model by using the tanh-coth method, etc.

However, no one has yet examined the GP model using the elegant auxiliary equation approach [24,41,42]. Therefore, motivated by the study of the aforementioned academics, our aim in this article is to search for scores of analytical solutions to the GP model utilizing the sophisticated auxiliary equation approach. The established solutions include periodic, kink, plane soliton, singular soliton, U shape, parabolic soliton, bell-shaped soliton etc. which are useful in plasma physics, optics, fluid mechanics and particle physics. The effects of dispersion coefficient α and free parameter κ on soliton pulses of the GP model have also been investigated.

Model description

In this sub-section, we considered the GP equation [29–40] which is read as.

$$u_t - \alpha u_{xxt} + 2\beta u_x - u u_{xxx} - \tau u u_x - \mu u_x u_{xx} = 0 \tag{1.1}$$

where α is the dispersion coefficient, β is the coefficient of the linearity and the steeping coefficients are τ and μ. Each coefficient is nonzero real numbers. This model includes many other nonlinear models, such as the Eq. (1.1) changes to Fornberg-Whitham equation [43], when $\alpha = 1, \tau = -1, \mu = 3$ and $\beta = 0.5$ to the Rosenau-Hyman equation [44], and to the Fuchssteiner-Fokas-Camassa-Holm equation when $\alpha = 0, \tau = 1, \mu = 3$ and $\beta = 0$.

Arrangement of the article

The remainder of the paper is laid out as follows: Section 2 provides an overview of the advanced auxiliary equation method. Section 3 discusses how to obtain wave solutions for the GP model. Section 4 discusses the graphical analysis of the derived solutions as well as the effects of the dispersion and free parameters on soliton pulses. Section 5 contains a comparison of the obtained solutions to earlier literature. In Section 6, we come to the conclusion.

Wave solutions of the GP equation

Eq. (1.1) is transformed into a nonlinear equation through the wave transformation $u(x,t) = u(\xi)$ and $\xi = x \pm \sigma t$:

$$(2\beta - \sigma)u + (\alpha\sigma - u)u'' + \frac{1-\mu}{2}(u')^2 - \frac{\tau}{2}u^2 = 0 \tag{2.1}$$

where σ is the speed of the wave and the primes denote the derivatives with respect to ξ. Using balancing principle yields $N = 2$ and expending the supposed solution of the mentioned method, we attain:

$$u(\xi) = C_0 + C_1 a^{g(\xi)} + C_2 a^{2g(\xi)} \tag{2.2}$$

Herein C_0, C_1, and C_2 are constants to be determined subsequently and the function $g(\xi)$ satisfies the first order auxiliary equation $g'(\xi) = \frac{1}{\ln(a)}\left\{ \upsilon a^{-g(\xi)} + \gamma + \kappa a^{g(\xi)} \right\}$. Substituting solution (2.2) into the Eq. (2.1), we get the values of the parameters as:

Set 1:

$\tau = -\gamma^2 + 4\kappa\upsilon, \mu = -3, \sigma = -\frac{2\beta}{\alpha\gamma^2 - 4\alpha\kappa\upsilon - 1}, C_0 = \frac{2\left(\pm\gamma^2 - \sqrt{\gamma^4 - 4\gamma^2\kappa\upsilon}\right)\beta\alpha}{\sqrt{\gamma^4 - 4\gamma^2\kappa\upsilon}(\alpha\gamma^2 - 4\alpha\kappa\upsilon - 1)}$,

$C_1 = \frac{4\left(\pm\frac{\gamma^2 - \sqrt{\gamma^4 - 4\gamma^2\kappa\upsilon}}{\sqrt{\gamma^4 - 4\gamma^2\kappa\upsilon}} + 1\right)\beta\alpha\kappa}{\gamma(\alpha\gamma^2 - 4\alpha\kappa\upsilon - 1)}, C_2 = 0.$

Therefore, the wave solutions of (1.1) are in the ensuing form:

When $\gamma^2 - 4\kappa\upsilon < 0$ and $\kappa \neq 0$,

$$u_{1,2}(x,t) = \frac{2\beta\alpha\left(\pm\sqrt{-\gamma^2 + 4\kappa\upsilon}\tan\left(\frac{\sqrt{-\gamma^2 + 4\kappa\upsilon}\,(\alpha\gamma^2 x - 4\alpha\kappa\upsilon x + 2\beta t - x)}{2(\alpha\gamma^2 - 4\alpha\kappa\upsilon - 1)}\right) - \sqrt{\gamma^2 - 4\kappa\upsilon}\right)}{\sqrt{\gamma^2 - 4\kappa\upsilon}(\alpha\gamma^2 - 4\alpha\kappa\upsilon - 1)} \tag{2.3}$$

$$u_{3,4}(x,t) = \frac{2\beta\alpha\left(\pm\sqrt{-\gamma^2 + 4\kappa\upsilon}\cot\left(\frac{\sqrt{-\gamma^2 + 4\kappa\upsilon}\,(\alpha\gamma^2 x - 4\alpha\kappa\upsilon x + 2\beta t - x)}{2(\alpha\gamma^2 - 4\alpha\kappa\upsilon - 1)}\right) - \sqrt{\gamma^2 - 4\kappa\upsilon}\right)}{\sqrt{\gamma^2 - 4\kappa\upsilon}(\alpha\gamma^2 - 4\alpha\kappa\upsilon - 1)} \tag{2.4}$$

When $\gamma^2 - 4\kappa\upsilon > 0$ and $\kappa \neq 0$,

$$u_{5,6}(x,t) = \frac{2\beta\alpha\left(\pm\tanh\left(\frac{\sqrt{\gamma^2 - 4\kappa\upsilon}\,(\alpha\gamma^2 x - 4\alpha\kappa\upsilon x + 2\beta t - x)}{2(\alpha\gamma^2 - 4\alpha\kappa\upsilon - 1)}\right) - 1\right)}{\alpha\gamma^2 - 4\alpha\kappa\upsilon - 1} \tag{2.5}$$

$$u_{7,8}(x,t) = \frac{2\beta\alpha\left(\pm\coth\left(\frac{\sqrt{\gamma^2 - 4\kappa\upsilon}\,(\alpha\gamma^2 x - 4\alpha\kappa\upsilon x + 2\beta t - x)}{2(\alpha\gamma^2 - 4\alpha\kappa\upsilon - 1)}\right) - 1\right)}{\alpha\gamma^2 - 4\alpha\kappa\upsilon - 1} \tag{2.6}$$

When $\gamma^2 + 4\upsilon^2 > 0, \kappa \neq 0$ and $\kappa = -\upsilon$,

$$u_{9,10}(x,t) = -\frac{2\beta\alpha\left(\pm\tanh\left(\frac{(\alpha(\gamma^2 + 4\upsilon^2)x + 2\beta t - x)\sqrt{\gamma^2 + 4\upsilon^2}}{-2 + (2\gamma^2 + 8\upsilon^2)\alpha}\right) - 1\right)}{-1 + (\gamma^2 + 4\upsilon^2)\alpha} \tag{2.7}$$

$$u_{11,12}(x,t) = -\frac{2\beta\alpha\left(\pm\coth\left(\frac{(\alpha(\gamma^2 + 4\upsilon^2)x + 2\beta t - x)\sqrt{\gamma^2 + 4\upsilon^2}}{-2 + (2\gamma^2 + 8\upsilon^2)\alpha}\right) + 1\right)}{-1 + (\gamma^2 + 4\upsilon^2)\alpha} \tag{2.8}$$

When $\gamma^2 + 4\upsilon^2 < 0, \kappa \neq 0$ and $\kappa = -\upsilon$,

$$u_{13,14}(x,t) = \frac{2\beta\alpha\left(\pm\sqrt{-\gamma^2 - 4\upsilon^2}\tan\left(\frac{(\alpha(\gamma^2 + 4\upsilon^2)x + 2\beta t - x)\sqrt{-\gamma^2 - 4\upsilon^2}}{-2 + (2\gamma^2 + 8\upsilon^2)\alpha}\right) - \sqrt{\gamma^2 + 4\upsilon^2}\right)}{\sqrt{\gamma^2 + 4\upsilon^2}(-1 + (\gamma^2 + 4\upsilon^2)\alpha)} \tag{2.9}$$

$$u_{15,16}(x,t) = \frac{2\beta\alpha\left(\pm\sqrt{-\gamma^2 - 4\upsilon^2}\cot\left(\frac{(\alpha(\gamma^2 + 4\upsilon^2)x + 2\beta t - x)\sqrt{-\gamma^2 - 4\upsilon^2}}{-2 + (2\gamma^2 + 8\upsilon^2)\alpha}\right) - \sqrt{\gamma^2 + 4\upsilon^2}\right)}{\sqrt{\gamma^2 + 4\upsilon^2}(-1 + (\gamma^2 + 4\upsilon^2)\alpha)} \tag{2.10}$$

When $\gamma^2 + 4\upsilon^2 < 0$ and $\kappa = \upsilon$,

$$u_{17,18}(x,t) = \frac{2\beta\alpha\left(\pm\sqrt{-\gamma^2 + 4\upsilon^2}\tan\left(\frac{\sqrt{-\gamma^2 + 4\upsilon^2}\,(\alpha\gamma^2 x - 4\alpha\upsilon^2 x + 2\beta t - x)}{2(\alpha\gamma^2 - 4\alpha\upsilon^2 - 1)}\right) - \sqrt{\gamma^2 - 4\upsilon^2}\right)}{\sqrt{\gamma^2 - 4\upsilon^2}(\alpha\gamma^2 - 4\alpha\upsilon^2 - 1)} \tag{2.11}$$

$$u_{19,20}(x,t) = \frac{2\beta\alpha\left(\pm\sqrt{-\gamma^2 + 4\upsilon^2}\cot\left(\frac{\sqrt{-\gamma^2 + 4\upsilon^2}\,(\alpha\gamma^2 x - 4\alpha\upsilon^2 x + 2\beta t - x)}{2(\alpha\gamma^2 - 4\alpha\upsilon^2 - 1)}\right) - \sqrt{\gamma^2 - 4\upsilon^2}\right)}{\sqrt{\gamma^2 - 4\upsilon^2}(\alpha\gamma^2 - 4\alpha\upsilon^2 - 1)} \tag{2.12}$$

When $\gamma^2 + 4\upsilon^2 > 0$ and $\kappa = \upsilon$,

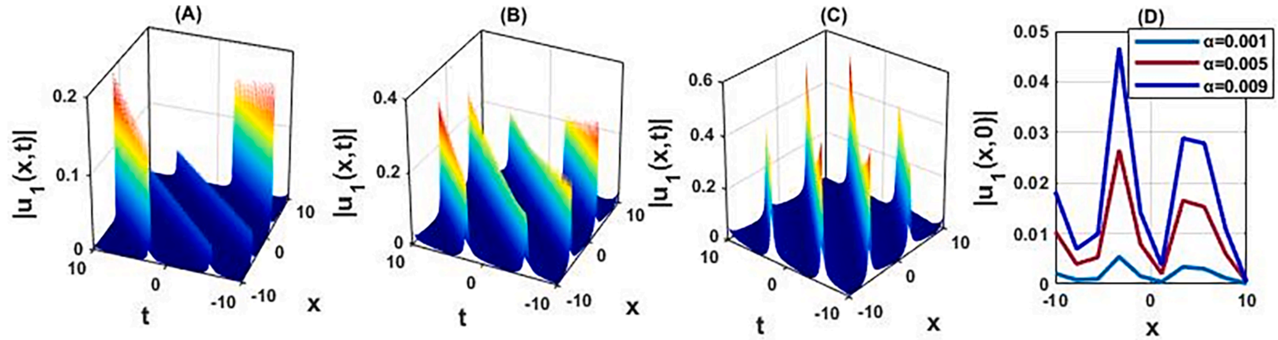

Fig. 1. 3D plots of the solution $|u_1(x,t)|$, for picking parameter values $\beta = 0.5, \gamma = 1, \kappa = 0.1, \upsilon = 1.01$ (A) for $\alpha = 0.001$, (B) for $\alpha = 0.005$, (C) for $\alpha = 0.009$ and (D) the 2D combined line plots of the solution $|u_1(x,t)|$ of (A)-(C) at $t = 0$.

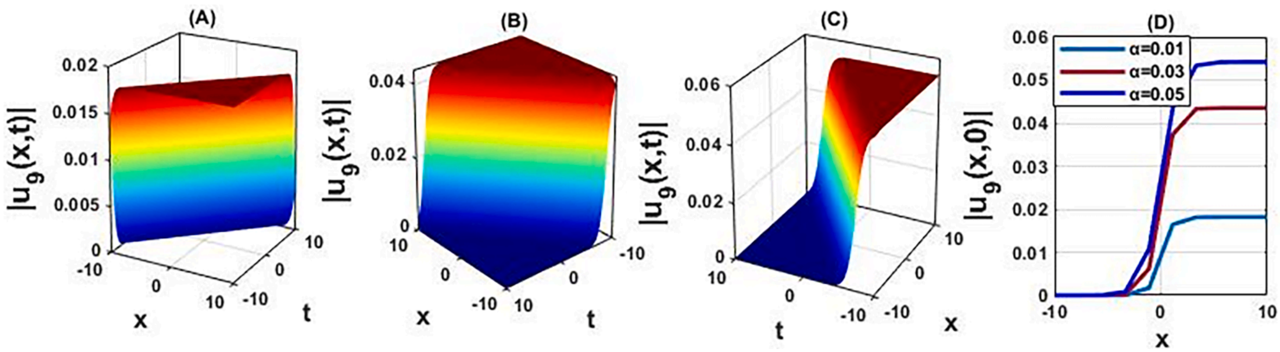

Fig. 2. 3D plots of the solutions $|u_9(x,t)|$ for the parameter values $\beta = 0.5, \gamma = 1, \upsilon = 1.01$ (A) for $\alpha = 0.01$, (B) for $\alpha = 0.03$, (C) for $\alpha = 0.05$ and (D) the 2D combined line plots of the solutions $|u_9(x,t)|$ of (A)-(C) at $t = 0$.

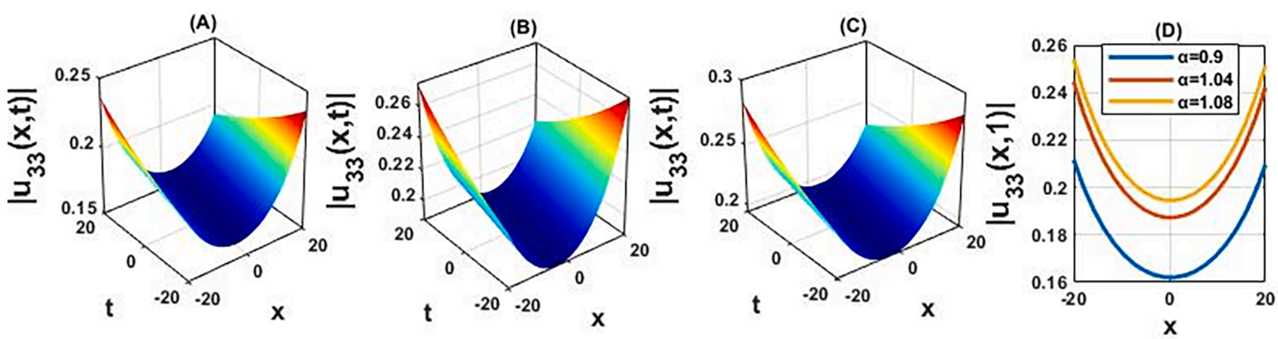

Fig. 3. 3D plots of the solutions $|u_{33}(x,t)|$ for parameter values $\beta = 0.09, \upsilon = 0.01$ (A) for $\alpha = 0.9$, (B) for $\alpha = 1.04$, (C) for $\alpha = 1.08$ and in Fig. (D), the 2D combined line plots of the solution $|u_{33}(x,t)|$ of (A)-(C) at $t = 1$.

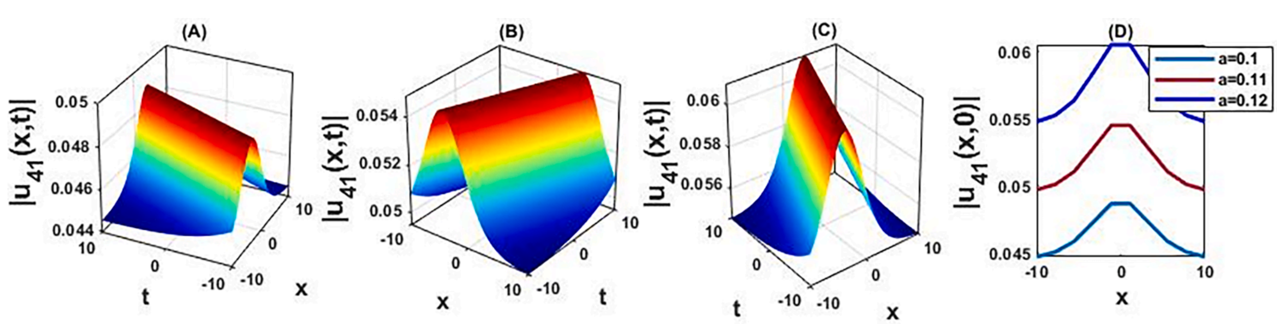

Fig. 4. 3D plots of the solutions $|u_{41}(x,t)|$ for picking parameters $\beta = 0.5, \gamma = 1, \kappa = 0.1, \upsilon = 1.01$ with (A) $\alpha = 0.1$, (B) $\alpha = 0.11$, (C) $\alpha = 0.12$ and (D) the 2D combined line plots of the solutions $|u_{41}(x,t)|$ of (A)-(C) at $t = 0$.

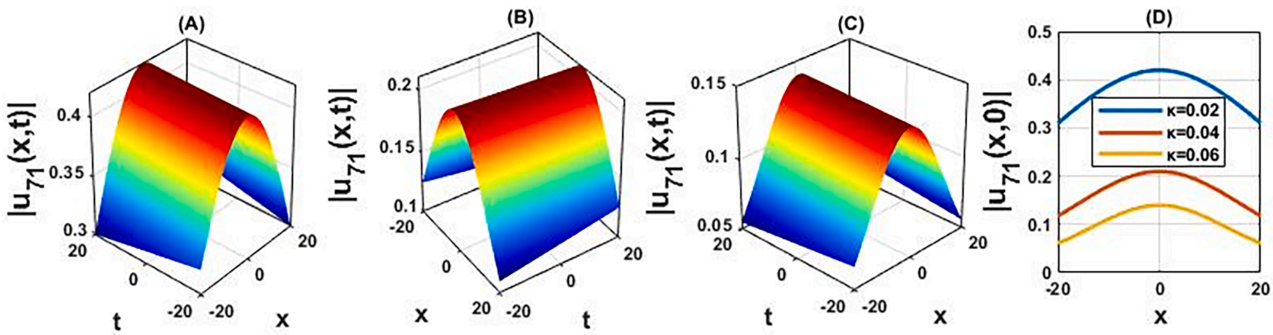

Fig. 5. 3D plots of the solutions $|u_{71}(x,t)|$ for $\beta = 0.5, \gamma = 1, \upsilon = 1.01$ with (A) $\kappa = 0.02$, (B) $\alpha = 0.04$, (C) $\alpha = 0.06$ and (D), the 2D combined line plots of (A)-(C) at $t = 0$.

Table 1
Comparison of the attained solutions with Yokus et al., [36] solutions.

Yokus et al., [36] solutions obtained by $(G'/G, 1/G)$	The solutions attained in this article
Taking $k = 1, c_1 = 0, \Omega = \dfrac{6\chi^2\epsilon}{(\chi^2-\alpha)(\chi^2\epsilon-1)}$, $\Gamma = \dfrac{\chi}{2}$ and $\Theta_{1,3}(x,t) = u(x,t)$, then the solutions of the Eq. (48) turns to $u(x,t) = \Omega\,\mathrm{cosech}^2\{\Gamma(x-\eta t)\}$	Taking $\Omega = -\dfrac{\upsilon C_2}{\kappa}, \Gamma = \sqrt{-\kappa\upsilon}, \eta = \dfrac{2\beta}{4\alpha\kappa\upsilon+1}$ and $u_{72}(x,t) = u(x,t)$, then the solutions of the Eq. (2.58) turns to $u(x,t) = \Omega\,\mathrm{cosech}^2\{\Gamma(x-\eta t)\}$
Choosing $k = 1, c_1 = 0, \Omega = \dfrac{6\chi^2\epsilon}{(\chi^2-\alpha)(\chi^2\epsilon-1)}, \Gamma = \dfrac{\chi}{2}$ and $\Theta_{2,3}(x,t) = u(x,t)$, then the solutions of the Eq. (53) turns to $u(x,t) = \Omega\mathrm{cosech}^2\{\Gamma(x-\eta t)\}$	Choosing $\Omega = -\dfrac{\upsilon C_2}{\kappa}, \Gamma = \sqrt{-\kappa\upsilon}, \eta = \dfrac{2\beta}{4\alpha\kappa\upsilon+1}$ and $u_{72}(x,t) = u(x,t)$, then the solutions of the Eq. (2.58) turns to $u(x,t) = \Omega\,\mathrm{cosech}^2\{\Gamma(x-\eta t)\}$
Inserting $k = 1, c_1 = 0, \Omega = \dfrac{24\epsilon}{\alpha}$ and $\Theta_{1,5}(x,t) = \Theta_{2,5}(x,t) = u(x,t)$, then the solutions of the Eqs. (50) and (55) turns to $u(x,t) = \dfrac{\Omega}{(x-\eta t)^2}$	Inserting $\Omega = \dfrac{C_2}{\kappa^2}, \eta = 2\beta$ and $u_{76}(x,t) = u(x,t)$, then the solutions of the Eq. (2.64) turns to $u(x,t) = \dfrac{\Omega}{(x-\eta t)^2}$

$$u_{21,22}(x,t) = \frac{2\beta\alpha\left(\pm\tanh\left(\frac{\sqrt{\gamma^2-4\upsilon^2}\,(\alpha\gamma^2x-4\alpha\upsilon^2x+2\beta t-x)}{2(\alpha\gamma^2-4\alpha\upsilon^2-1)}\right)-1\right)}{\alpha\gamma^2-4\alpha\upsilon^2-1} \tag{2.13}$$

$$u_{23,24}(x,t) = \frac{2\beta\alpha\left(\pm\coth\left(\frac{\sqrt{\gamma^2-4\upsilon^2}\,(\alpha\gamma^2x-4\alpha\upsilon^2x+2\beta t-x)}{2(\alpha\gamma^2-4\alpha\upsilon^2-1)}\right)-1\right)}{\alpha\gamma^2-4\alpha\upsilon^2-1} \tag{2.14}$$

When $\gamma = \kappa = \varphi$ and $\upsilon = 0$,

$$u_{25,26}(x,t) = \pm\frac{2\exp\left(\varphi\left(\frac{2\beta t}{\alpha\varphi^2-1}+x\right)\right)\beta\alpha}{\left(1-exp\left(\varphi\left(\frac{2\beta t}{\alpha\varphi^2-1}+x\right)\right)\right)(\alpha\varphi^2-1)} \tag{2.15}$$

When $\gamma = \kappa + \upsilon$,

$$u_{27,28}(x,t) = \pm\frac{2\beta\alpha\left(\exp\left(\frac{2\left(\frac{x(-\kappa+\upsilon)^2\alpha}{2}+\beta t-\frac{x}{2}\right)(-\kappa+\upsilon)}{-1+(-\kappa+\upsilon)^2\alpha}\right)\kappa+1\right)}{\left(\exp\left(\frac{2\left(\frac{x(-\kappa+\upsilon)^2\alpha}{2}+\beta t-\frac{x}{2}\right)(-\kappa+\upsilon)}{-1+(-\kappa+\upsilon)^2\alpha}\right)\kappa-1\right)(\alpha\kappa^2-2\alpha\kappa\upsilon+\alpha\upsilon^2-1)} \tag{2.16}$$

When $\gamma = -(\kappa+\upsilon)$,

$$u_{29,30}(x,t) = \pm\frac{2\beta\alpha\left(\exp\left(\frac{2\left(\frac{x(-\kappa+\upsilon)^2\alpha}{2}+\beta t-\frac{x}{2}\right)(-\kappa+\upsilon)}{-1+(-\kappa+\upsilon)^2\alpha}\right)+\kappa\right)}{\left(\kappa-\exp\left(\frac{2\left(\frac{x(-\kappa+\upsilon)^2\alpha}{2}+\beta t-\frac{x}{2}\right)(-\kappa+\upsilon)}{-1+(-\kappa+\upsilon)^2\alpha}\right)\right)\left(-1+(-\kappa+\upsilon)^2\alpha\right)} \tag{2.17}$$

When $\upsilon = 0$,

$$u_{31,32}(x,t) = \frac{2\beta\alpha\left(\pm\kappa\exp\left(\frac{\gamma(\alpha\gamma^2x+2\beta t-x)}{\alpha\gamma^2-1}\right)-1\right)}{\left(\kappa\exp\left(\frac{\gamma(\alpha\gamma^2x+2\beta t-x)}{\alpha\gamma^2-1}\right)-1\right)(\alpha\gamma^2-1)} \tag{2.18}$$

When $\kappa = \gamma = \upsilon \neq 0$,

$$u_{33,34}(x,t) = \frac{2\beta\alpha\left(\pm\tan\left(\frac{\sqrt{3}\,\upsilon(3\alpha\upsilon^2x-2\beta t+x)}{2(3\alpha\upsilon^2+1)}\right)\upsilon^2+\sqrt{-\upsilon^4}\right)}{\sqrt{-\upsilon^4}(3\alpha\upsilon^2+1)} \tag{2.19}$$

When $\upsilon = \gamma = \varphi$ and $\kappa = 0$; and when $\kappa = 0$ inputting the values results in constant solutions, which are not mentioned here because constant solutions have no physical relevance.

Set 2:

$$\tau = \frac{2(\gamma^4-8\gamma^2K\upsilon+16\upsilon^2K^2)\alpha}{\alpha\gamma^2-4\alpha K\upsilon+1},\ \mu = -2,\ \sigma = \frac{2(2\alpha\gamma^2-8\alpha K\upsilon+1)\beta}{\alpha^2\gamma^4-8\alpha^2\gamma^2K\upsilon+16\alpha^2K^2\upsilon^2+2\alpha\gamma^2-8\alpha K\upsilon+1},$$

Table 2
Comparison of the attained solutions with Bilal et al., [39] solutions.

Bilal et al., [39] solutions attained by (G'/G^2)-expansion method	The solutions attained in this article
Taking $\Upsilon = -1, \Gamma = 1, \Delta = \dfrac{3}{4}(2k-\eta), \Omega = -\dfrac{1}{2}(2k-\eta)$ and $\Theta_{2,2}(x,t) = u(x,t)$, then the solutions of the Eq. (27) turns to $u(x,t) = \Delta\tanh^2(x-\eta t) + \Omega$	Taking $\Delta = \dfrac{24\alpha^2\beta}{16\alpha^2-1}, \Omega = \dfrac{2\alpha\beta(-8\alpha-1)}{16\alpha^2-1}, \upsilon = -1$, $\kappa = 1, \eta = \dfrac{2\beta(8\alpha+1)}{16\alpha^2+8\alpha+1}$ and $u_{48}(x,t) = u(x,t)$, then the solutions of the Eq. (2.33) turns to $u(x,t) = \Delta\tanh^2(x-\eta t) + \Omega$
Selecting $\Delta = \dfrac{3(2k-\eta)}{16\Upsilon\Gamma}, \Omega = \dfrac{2(2k-\eta)}{16\Upsilon\Gamma}, \chi = 2\sqrt{\Upsilon\Gamma}$ and $\Theta_{3,1}(x,t) = u(x,t)$, then the solutions of the Eq. (30) turns to $u(x,t) = \Delta\tan^2\{\chi(x-\eta t)\} + \Omega$	Selecting $\Delta = \dfrac{24\alpha^2\beta\upsilon^2}{9\alpha^2\upsilon^2-1}, \Omega = \dfrac{2\alpha\beta(-3\alpha\upsilon^2-1)}{9\alpha^2\upsilon^4-1}$, $\chi = \dfrac{\sqrt{3}\upsilon(9\alpha^2\upsilon^4-6\alpha\upsilon^2+1)}{2(3\alpha\upsilon^2-1)}, \eta = \dfrac{2\beta-12\alpha\beta\upsilon^2}{9\alpha^2\upsilon^4-6\alpha\upsilon^2+1}$ and $u_{55}(x,t) = u(x,t)$, then the solutions of the Eq. (2.40) turns to $u(x,t) = \Delta\tan^2\{\chi(x-\eta t)\} + \Omega$

$$C_0 = \frac{2\beta\alpha(\alpha\gamma^2 + 8\alpha\upsilon\kappa - 1)}{\alpha^2\gamma^4 - 8\alpha^2\gamma^2\kappa\upsilon + 16\alpha^2\kappa^2\upsilon^2 - 1}, C_1 = \frac{24\gamma\beta\kappa\alpha^2}{\alpha^2\gamma^4 - 8\alpha^2\gamma^2\kappa\upsilon + 16\alpha^2\kappa^2\upsilon^2 - 1},$$

$$C_2 = \frac{24\beta\kappa^2\alpha^2}{\alpha^2\gamma^4 - 8\alpha^2\gamma^2\kappa\upsilon + 16\alpha^2\kappa^2\upsilon^2 - 1}.$$

Therefore, the wave solutions of (1.1) are established in the following form:

When $\gamma^2 - 4\kappa\upsilon < 0$ and $\upsilon \neq 0$,

$$u_{35}(x,t) = \frac{2\beta\alpha\left(-1 + (-\gamma^2 + 4\kappa\upsilon)\left(3\tan(\chi)^2 + 2\right)\alpha\right)}{-1 + 16\left(\kappa\upsilon - \frac{\gamma^2}{4}\right)^2\alpha^2} \tag{2.20}$$

$$u_{36}(x,t) = \frac{2\beta\alpha\left(-1 + (-\gamma^2 + 4\kappa\upsilon)\left(3\cot(\chi)^2 + 2\right)\alpha\right)}{-1 + 16\left(\kappa\upsilon - \frac{\gamma^2}{4}\right)^2\alpha^2} \tag{2.21}$$

where $\chi = \frac{8\left(\left(\kappa\upsilon - \frac{\gamma^2}{4}\right)^2\alpha^2 x + \left(\beta t - \frac{x}{2}\right)\left(\kappa\upsilon - \frac{\gamma^2}{4}\right)\alpha - \frac{\beta t}{8} + \frac{x}{16}\right)\sqrt{-\gamma^2 + 4\kappa\upsilon}}{\left(-\alpha\gamma^2 + 4\alpha\kappa\upsilon - 1\right)^2}$.

When $\gamma^2 - 4\kappa\upsilon > 0$ and $\upsilon \neq 0$,

$$u_{37}(x,t) = -\frac{2\beta\alpha\left(1 + (-\gamma^2 + 4\kappa\upsilon)\left(3\tanh(\chi)^2 - 2\right)\alpha\right)}{-1 + 16\left(\kappa\upsilon - \frac{\gamma^2}{4}\right)^2\alpha^2} \tag{2.22}$$

$$u_{38}(x,t) = -\frac{2\beta\alpha\left(1 + (-\gamma^2 + 4\kappa\upsilon)\left(3\coth(\chi)^2 - 2\right)\alpha\right)}{-1 + 16\left(\kappa\upsilon - \frac{\gamma^2}{4}\right)^2\alpha^2} \tag{2.23}$$

When $\gamma^2 + 4\upsilon^2 < 0, \kappa \neq 0$ and $\kappa = -\upsilon$,

$$u_{39}(x,t) = -\frac{2\beta\alpha\left(1 + (\gamma^2 + 4\upsilon^2)\left(3\tan(\chi)^2 + 2\right)\alpha\right)}{-1 + 16\left(\upsilon^2 + \frac{\gamma^2}{4}\right)^2\alpha^2} \tag{2.24}$$

$$u_{40}(x,t) = -\frac{2\beta\alpha\left(1 + (\gamma^2 + 4\upsilon^2)\left(3\cot(\chi)^2 + 2\right)\alpha\right)}{-1 + 16\left(\upsilon^2 + \frac{\gamma^2}{4}\right)^2\alpha^2} \tag{2.25}$$

where $\chi = \frac{8\left(-\left(\upsilon^2 + \frac{\gamma^2}{4}\right)^2\alpha^2 x + \left(\beta t - \frac{x}{2}\right)\left(\upsilon^2 + \frac{\gamma^2}{4}\right)\alpha + \frac{\beta t}{8} - \frac{x}{16}\right)\sqrt{-\gamma^2 - 4\upsilon^2}}{\left(\alpha\gamma^2 + 4\alpha\upsilon^2 + 1\right)^2}$.

When $\gamma^2 + 4\upsilon^2 > 0, \kappa \neq 0$ and $\kappa = -\upsilon$,

$$u_{41}(x,t) = \frac{2\beta\alpha\left(-1 + (\gamma^2 + 4\upsilon^2)\left(3\tanh(\chi)^2 - 2\right)\alpha\right)}{-1 + 16\left(\upsilon^2 + \frac{\gamma^2}{4}\right)^2\alpha^2} \tag{2.26}$$

$$u_{42}(x,t) = \frac{2\beta\alpha\left(-1 + (\gamma^2 + 4\upsilon^2)\left(3\coth(\chi)^2 - 2\right)\alpha\right)}{-1 + 16\left(\upsilon^2 + \frac{\gamma^2}{4}\right)^2\alpha^2} \tag{2.27}$$

When $\gamma^2 + 4\upsilon^2 < 0$ and $\kappa = \upsilon$,

$$u_{43}(x,t) = \frac{2\beta\alpha\left(-\alpha\left(3\tan(\chi)^2 + 2\right)\gamma^2 - 1 + 4\alpha\left(\left(3\tan(\chi)^2 + 2\right)\upsilon^2\right)\right)}{\alpha^2\gamma^4 - 8\alpha^2\gamma^2\upsilon^2 + 16\alpha^2\upsilon^4 - 1} \tag{2.28}$$

$$u_{44}(x,t) = \frac{2\beta\alpha\left(-\alpha\left(3\cot(\chi)^2 + 2\right)\gamma^2 - 1 + 4\alpha\left(\left(3\cot(\chi)^2 + 2\right)\upsilon^2\right)\right)}{\alpha^2\gamma^4 - 8\alpha^2\gamma^2\upsilon^2 + 16\alpha^2\upsilon^4 - 1} \tag{2.29}$$

where $\chi = \frac{8\left(x\left(\upsilon + \frac{\gamma}{2}\right)^2\left(\upsilon - \frac{\gamma}{2}\right)^2\alpha^2 + \left(\beta t - \frac{x}{2}\right)\left(\upsilon + \frac{\gamma}{2}\right)\left(\upsilon - \frac{\gamma}{2}\right)\alpha - \frac{\beta t}{8} + \frac{x}{16}\right)\sqrt{-\gamma^2 + 4\upsilon^2}}{\left(-\alpha\gamma^2 + 4\alpha\upsilon^2 - 1\right)^2}$.

When $\gamma^2 + 4\upsilon^2 > 0$ and $\kappa = \upsilon$,

$$u_{45}(x,t) = -\frac{2\beta\alpha\left(-\alpha\left(3\tanh(\chi)^2 - 2\right)\gamma^2 + 1 + 4\alpha\left(3\tanh(\chi)^2 - 2\right)\upsilon^2\right)}{\alpha^2\gamma^4 - 8\alpha^2\gamma^2\upsilon^2 + 16\alpha^2\upsilon^{4'} - 1} \tag{2.30}$$

$$u_{46}(x,t) = -\frac{2\beta\alpha\left(-\alpha\left(3\tanh(\chi)^2 - 2\right)\gamma^2 + 1 + 4\alpha\left(3\tanh(\chi)^2 - 2\right)\upsilon^2\right)}{\alpha^2\gamma^4 - 8\alpha^2\gamma^2\nu^2 + 16\alpha^2\nu^{4'} - 1} \tag{2.31}$$

where $\chi = \frac{8(\left(\nu - \frac{\gamma}{2}\right)^2\left(\nu + \frac{\gamma}{2}\right)^2 x\,\alpha^2 + \left(\beta t - \frac{x}{2}\right)\left(\nu + \frac{\gamma}{2}\right)\left(\nu - \frac{\gamma}{2}\right)\alpha - \frac{\beta t}{8} + \frac{x}{16})\sqrt{\gamma^2 - 4\nu^2})}{\left(-\alpha\gamma^2 + 4\alpha\nu^2 - 1\right)^2}$.

When $\gamma^2 = 4\kappa\upsilon$,

$$u_{47}(x,t) = \frac{8\beta\alpha\left(\beta^2t^2 - \beta tx + \frac{x^2}{4} - 3\alpha\right)}{(2\beta t - x)^2} \tag{2.32}$$

When $\kappa\upsilon < 0$, $\gamma = 0$ and $\kappa \neq 0$,

$$u_{48}(x,t) = -\frac{2\beta\alpha\left(12\upsilon\kappa\tanh(\chi)^2\alpha - 8\alpha\upsilon\kappa + 1\right)}{16\alpha^2\kappa^2\upsilon^2 - 1} \tag{2.33}$$

$$u_{49}(x,t) = -\frac{2\beta\alpha\left(12\upsilon\kappa\coth(\chi)^2\alpha - 8\alpha\upsilon\kappa + 1\right)}{16\alpha^2\kappa^2\upsilon^2 - 1} \tag{2.34}$$

where$\chi = \frac{\sqrt{-\kappa\upsilon}\left(16\alpha^2\kappa^2\upsilon^2 x + 16\alpha\beta\kappa\upsilon t - 8\alpha\upsilon\kappa x - 2\beta t + x\right)}{\left(4\alpha\kappa\upsilon - 1\right)^2}$.

When $\gamma = 0$ and $\upsilon = -\kappa$,

$$u_{50}(x,t) = \frac{24(1 + \exp(-2\kappa(\chi)))^2\beta\alpha^2\kappa^2}{(-1 + \exp(-2\kappa(\chi)))^2(16\alpha^2\kappa^4 - 1)} + \frac{2\beta\alpha(-8\alpha\kappa^2 - 1)}{16\alpha^2\kappa^4 - 1} \tag{2.35}$$

where $\chi = -\frac{2\left(8\alpha\kappa^2 + 1\right)\beta t}{16\alpha^2\kappa^4 + 8\alpha\kappa^2 + 1} + x$.

When $\gamma = \kappa = \varphi$ and $\upsilon = 0$,

$$u_{51}(x,t) = \frac{24(\exp(\varphi(\chi)))^2\beta\alpha^2\varphi^2}{(1 - \exp(\varphi(\chi)))^2(\alpha^2\varphi^4 - 1)} + \frac{24\exp(\varphi(\chi))\beta\alpha^2\varphi^2}{(1 - \exp(\varphi(\chi)))(\alpha^2\varphi^4 - 1)} + \frac{2\beta\alpha(\alpha\varphi^2 - 1)}{\alpha^2\varphi^4 - 1} \tag{2.36}$$

where $\chi = -\frac{2\left(2\alpha\varphi^2 + 1\right)\beta t}{\alpha^2\varphi^4 + 2\alpha\varphi^2 + 1} + x$.

When $\gamma = \kappa + \upsilon$,

$$u_{52}(x,t) = \frac{2\beta\alpha\left(10\left(\frac{1}{5} + (\upsilon - \kappa)^2\alpha\right)\kappa\exp(\chi) + \left(-1 + (\upsilon - \kappa)^2\alpha\right)\left(\exp(\psi)\kappa^2 + 1\right)\right)}{\left(-1 + (\nu - \kappa)^2\alpha\right)\left(\kappa\exp(\psi) - 1\right)^2\left(1 + (\nu - \kappa)^2\alpha\right)} \tag{2.37}$$

where$\chi = -\frac{4(\nu - \kappa)\left(-\frac{x(\nu - \kappa)^4\alpha^2}{4} + \left(\beta t - \frac{x}{2}\right)(\nu - \kappa)^2\alpha + \frac{\beta t}{2} - \frac{x}{4}\right)}{\left(1 + (\nu - \kappa)^2\alpha\right)^2}$,

and$\psi = -\frac{8(\nu - \kappa)\left(-\frac{x(\nu - \kappa)^4\alpha^2}{4} + \left(\beta t - \frac{x}{2}\right)(\nu - \kappa)^2\alpha + \frac{\beta t}{2} - \frac{x}{4}\right)}{\left(1 + (\nu - \kappa)^2\alpha\right)^2}$.

When$\gamma = -(\kappa + \upsilon)$,

$$u_{53}(x,t) = \frac{2\beta\alpha\left(10\left(\frac{1}{5} + (\upsilon - \kappa)^2\alpha\right)\kappa\exp(\chi) + \left(-1 + (\upsilon - \kappa)^2\alpha\right)\left(\exp(\psi) + \kappa^2\right)\right)}{\left(1 + (\nu - \kappa)^2\alpha\right)\left(-1 + (\nu - \kappa)^2\alpha\right)\left(\kappa - \exp(\chi)\right)^2} \tag{2.38}$$

where$\chi = -\frac{4(\nu - \kappa)\left(-\frac{x(\nu - \kappa)^4\alpha^2}{4} + \left(\beta t - \frac{x}{2}\right)(\nu - \kappa)^2\alpha + \frac{\beta t}{2} - \frac{x}{4}\right)}{\left(1 + (\nu - \kappa)^2\alpha\right)^2}$,

and$\psi = -\frac{8(\nu - \kappa)\left(-\frac{x(\nu - \kappa)^4\alpha^2}{4} + \left(\beta t - \frac{x}{2}\right)(\nu - \kappa)^2\alpha + \frac{\beta t}{2} - \frac{x}{4}\right)}{\left(1 + (\nu - \kappa)^2\alpha\right)^2}$.

When$\upsilon = 0$,

$$u_{54}(x,t) = \frac{2\alpha\beta(\kappa^2(\alpha\gamma^2 - 1)\exp(\chi) + (10\gamma^2\kappa\alpha + 2\kappa)\exp(\chi) + \alpha\gamma^2 - 1)}{(\kappa\exp(\chi) - 1)^2(\alpha^2\gamma^4 - 1)} \tag{2.39}$$

Where $\chi = \frac{\gamma\left(\alpha^2\gamma^4 x - 4\alpha\beta t\gamma^2 + 2\alpha\gamma^2 x - 2\beta t + x\right)}{\left(\alpha\gamma^2 + 1\right)^2}$.

When$\kappa = \gamma = \upsilon \neq 0$,

$$u_{55}(x,t) = \frac{2\beta\alpha\left(9\tan\left(\frac{\sqrt{3}\,\upsilon(9\alpha^2\upsilon^4 x + 12\alpha\beta\upsilon^2 t - 6\alpha\upsilon^2 x - 2\beta t + x}{2(3\alpha\upsilon^2 - 1)^2}\right)^2\alpha\upsilon^2 + 6\alpha\upsilon^2 - 1\right)}{9\alpha^2\upsilon^4 - 1} \tag{2.40}$$

When$\upsilon = \gamma = 0$,

$$u_{56}(x,t) = -\frac{24\beta\alpha^2}{(-2\beta t + x)^2} + 2\beta\alpha \tag{2.41}$$

When $\kappa = \upsilon$ and $\gamma = 0$,

$$u_{57}(x,t) = \frac{24tan^2\left(\upsilon\left(-\frac{2(-8\alpha\upsilon^2+1)\beta t}{16\alpha^2\upsilon^4-8\alpha\upsilon^2+1} + x\right)\right)\beta\alpha^2\upsilon^2}{16\alpha^2\upsilon^4-1} + \frac{2\beta\alpha(8\alpha\upsilon^2-1)}{16\alpha^2\upsilon^4-1} \tag{2.42}$$

When $\upsilon = \kappa = 0$; $\upsilon = \gamma = \varphi$ and $\kappa = 0$; $\kappa = \gamma = 0$; and $\kappa = 0$, entering the values lead to constant solutions which are not stated here since constant solutions have no physical implication.

Set 3:

$$\tau = \frac{\gamma^4\alpha C_2 - 8\gamma^2\alpha\kappa\upsilon C_2 + 16\alpha\upsilon^2\kappa^2 C_2 - 24\alpha\beta\kappa^2 - \gamma^2 C_2 + 4\kappa\upsilon C_2}{B_2(\alpha\gamma^2 - 4\alpha\kappa\upsilon - 1)}, \mu = -2$$
$$\sigma = -\frac{2\beta}{\alpha\gamma^2 - 4\alpha\kappa\upsilon - 1}, C_0 = \frac{\upsilon C_2}{\kappa}, C_2 = C_2, C_1 = \frac{\gamma C_2}{\kappa}.$$

Therefore, for the values presented in Set 3, the wave solutions of (1.1) are found in the form:

When $\gamma^2 - 4\kappa\upsilon < 0$ and $\upsilon \neq 0$,

$$u_{58}(x,t) = \frac{C_2\left(\tan\left(\frac{\sqrt{(-\gamma^2+4\kappa\upsilon}\left(\alpha\gamma^2 x - 4\alpha\kappa\upsilon x + 2\beta t - x\right)}{2(\alpha\gamma^2-4\alpha\kappa\upsilon-1)}\right)^2 + 1\right)\left(\kappa\upsilon - \frac{\gamma^2}{4}\right)}{\kappa^2} \tag{2.43}$$

$$u_{59}(x,t) = \frac{C_2\left(\cot\left(\frac{\sqrt{(-\gamma^2+4\kappa\upsilon}\left(\alpha\gamma^2\ x - 4\alpha\kappa\upsilon\ x + 2\beta t - x\right)}{2(\alpha\gamma^2-4\alpha\kappa\upsilon-1)}\right)^2 + 1\right)\left(\kappa\upsilon - \frac{\gamma^2}{4}\right)}{\kappa^2} \tag{2.44}$$

When $\gamma^2 - 4\kappa\upsilon > 0$ and $\upsilon \neq 0$,

$$u_{60}(x,t) = -\frac{\left(\kappa\nu - \frac{\gamma^2}{4}\right)(\tanh(\Omega)+1)(\tanh(\Omega)+1)C_2}{\kappa^2} \tag{2.45}$$

$$u_{61}(x,t) = -\frac{C_2\left(\kappa\nu - \frac{\gamma^2}{4}\right)(\coth(\Omega)-1)(\coth(\Omega)+1)}{\kappa^2} \tag{2.46}$$

where $\Omega = \frac{\sqrt{(\gamma^2-4\kappa\nu}\left(\alpha\gamma^2 x - 4\alpha\kappa\nu x + 2\beta t - x\right)}{2(\alpha\gamma^2-4\alpha\kappa\nu-1)}$.

when $\gamma^2 + 4\upsilon^2 < 0, \kappa \neq 0$ and $\kappa = -\upsilon$,

$$u_{62}(x,t) = -\frac{C_2\left(\upsilon^2 + \frac{\gamma^2}{4}\right)\left(\tan\left(\frac{(x(\gamma^2+4\upsilon^2)\alpha + 2\beta t - x)\sqrt{-\gamma^2-4\upsilon^2}}{-2(2\gamma^2+8\upsilon^2)\alpha}\right)^2 + 1\right)}{\upsilon^2} \tag{2.47}$$

$$u_{63}(x,t) = -\frac{C_2\left(\upsilon^2 + \frac{\gamma^2}{4}\right)\left(\cot\left(\frac{(x(\gamma^2+4\upsilon^2)\alpha + 2\beta t - x)\sqrt{-\gamma^2-4\upsilon^2}}{-2(2\gamma^2+8\upsilon^2)\alpha}\right)^2 + 1\right)}{\upsilon^2} \tag{2.48}$$

When $\gamma^2 + 4\upsilon^2 > 0, \kappa \neq 0$ and $\kappa = -\upsilon$,

$$u_{64}(x,t) = \frac{C_2\left(\upsilon^2 + \frac{\gamma^2}{4}\right)(\tanh(\Omega)-1)(\tanh(\Omega)+1)}{\nu^2} \tag{2.49}$$

$$u_{65}(x,t) = \frac{C_2\left(\upsilon^2 + \frac{\gamma^2}{4}\right)(\coth(\Omega)+1)(\coth(\Omega)-1)}{\nu^2} \tag{2.50}$$

where $\Omega = \frac{\left(x\left(\gamma^2+4\upsilon^2\right)\alpha + 2\beta t - x\right)\sqrt{\gamma^2+4\upsilon^2}}{-2+(2\gamma^2+8\upsilon^2)\alpha}$.

When $\gamma^2 + 4\upsilon^2 < 0$ and $\kappa = \upsilon$,

$$u_{66}(x,t) = \frac{C_2\left(\tan\left(\frac{\sqrt{(-\gamma^2+4\upsilon^2}\left(\alpha\gamma^2 x - 4\alpha\upsilon^2 x + 2\beta t - x\right)}{2(\alpha\gamma^2-4\alpha\upsilon^2-1)}\right)^2 + 1\right)\left(\upsilon - \frac{\gamma}{2}\right)\left(\upsilon + \frac{\gamma}{2}\right)}{\upsilon^2} \tag{2.51}$$

$$u_{67}(x,t) = \frac{C_2\left(\cot\left(\frac{\sqrt{(-\gamma^2+4\upsilon^2}\left(\alpha\gamma^2 x - 4\alpha\upsilon^2 x + 2\beta t - x\right)}{2(\alpha\gamma^2-4\alpha\upsilon^2-1)}\right)^2 + 1\right)\left(\upsilon + \frac{\gamma}{2}\right)\left(\upsilon - \frac{\gamma}{2}\right)}{\upsilon^2} \tag{2.52}$$

When $\gamma^2 + 4\upsilon^2 > 0$ and $\kappa = \upsilon$,

$$u_{68}(x,t) = -\frac{C_2\left(-\gamma^2\tanh^2(\chi) + 4\upsilon^2\tanh^2(\chi) + \gamma^2 - 4\upsilon^2\right)}{4\upsilon^2} \tag{2.53}$$

$$u_{69}(x,t) = -\frac{C_2\left(-\gamma^2\coth^2(\chi) + 4\upsilon^2\coth^2(\chi) + \gamma^2 - 4\upsilon^2\right)}{4\upsilon^2} \tag{2.54}$$

where $\chi = \frac{\sqrt{(\gamma^2-4\upsilon^2}\left(\alpha\gamma^2\ x - 4\alpha\upsilon^2\ x + 2\beta t - x\right)}{2(\alpha\gamma^2-4\alpha\upsilon^2-1)}$.

When $\gamma^2 = 4\kappa\upsilon$,

$$u_{70}(x,t) = \frac{\left(2\sqrt{\kappa\upsilon}(-2\beta t + x) + 2\right)^2 C_2}{4\kappa^2(-2\beta t + x)^2} - \frac{\left(2\sqrt{\kappa\nu}(-2\beta t + x) + 2\right)\sqrt{\kappa\upsilon} C_2}{\kappa^2(-2\beta t + x)} + \frac{\upsilon C_2}{\kappa} \tag{2.55}$$

When $\kappa\upsilon < 0, \gamma = 0$ and $\kappa \neq 0$,

$$u_{71}(x,t) = \frac{\upsilon C_2}{\kappa}\text{sech}^2\left(\frac{\sqrt{-\kappa\upsilon}\,(4\alpha\kappa\upsilon x - 2\beta t + x)}{4\alpha\kappa\upsilon + 1}\right) \tag{2.56}$$

$$u_{72}(x,t) = -\frac{\upsilon C_2}{\kappa}\text{cosech}^2\left(\frac{\sqrt{-\kappa\upsilon}\,(4\alpha\kappa\upsilon x - 2\beta t + x)}{4\alpha\kappa\upsilon + 1}\right) \tag{2.57}$$

When $\gamma = 0$ and $\upsilon = -\kappa$,

$$u_{73}(x,t) = \frac{4C_2\exp\left(\frac{-4\left(2\alpha\kappa^2 x + \beta t - \frac{1}{2}x\right)\kappa}{4\alpha\kappa^2-1}\right)}{\left(-1 + \exp\left(\frac{-4\left(2\alpha\kappa^2 x + \beta t - \frac{1}{2}x\right)\kappa}{4\alpha\kappa^2-1}\right)\right)^2} \tag{2.58}$$

When $\gamma = \kappa = \varphi$ and $\upsilon = 0$,

$$u_{74}(x,t) = \frac{C_2(\exp(\varphi(\chi)))^2}{(1-\exp(\varphi(\chi)))^2} + \frac{C_2\exp(\varphi(\chi))}{1-\exp(\varphi(\chi))} \tag{2.59}$$

where $\chi = \frac{2\beta t}{\alpha\varphi^2-1} + x$.

When $\gamma = \kappa + \upsilon$,

$$u_{75}(x,t) = \frac{C_2(\upsilon-\kappa)^2\exp(\chi)}{(\kappa\exp(\chi)-1)^2\kappa} \tag{2.60}$$

where $\chi = \frac{2(\upsilon-\kappa)\left(\frac{x\ (\upsilon-\kappa)^2\alpha}{2} + \beta\ t - \frac{x}{2}\right)}{-1+(\upsilon-\kappa)^2\alpha}$.

When $\gamma = -(\kappa + \upsilon)$,

$$u_{76}(x,t) = \frac{C_2(\upsilon-\kappa)^2\exp(\chi)}{(\kappa-\exp(\chi))^2\kappa} \tag{2.61}$$

where $\chi = \frac{2(\upsilon-\kappa)\left(\frac{x\ (\upsilon-\kappa)^2\alpha}{2} + \beta\ t - \frac{x}{2}\right)}{-1+(\upsilon-\kappa)^2\alpha}$.

When $\upsilon = 0$,

$$u_{74}(x,t) = \frac{C_2\gamma^2(\exp(\gamma(\chi)))^2}{(1-\kappa\exp(\gamma(\chi)))^2} + \frac{C_2\gamma^2\exp(\gamma(\chi))}{\kappa(1-\kappa\exp(\gamma(\chi)))} \tag{2.62}$$

where $\chi = \frac{2\beta\ t}{\alpha\gamma^2-1} + x$.

When $\kappa = \gamma = \upsilon \neq 0$,

$$u_{75}(x,t) = \frac{3C_2}{4}\sec^2\left(\frac{\sqrt{3}\,\upsilon(3\alpha\upsilon^2 x - 2\beta t + x)}{2(3\alpha\upsilon^2+1)}\right) \tag{2.63}$$

When $\upsilon = \gamma = 0$,

$$u_{76}(x,t) = \frac{C_2}{\kappa^2(-2\beta t + x)^2} \tag{2.64}$$

When $\kappa = \upsilon$ and $\gamma = 0$,

$$u_{77}(x,t) = C_2\tan^2\left(\upsilon\left(\frac{2\beta t}{-4\alpha\upsilon^2 - 1} + x\right)\right) + C_2 \tag{2.65}$$

Inserting the values results in constant solutions when $\kappa = 0$, which are not given here because they are not applicable in real fields.

Results and discussion

In the Gilson-Pickering equation, α is the dispersion coefficient, β is the linear coefficient and the steeping coefficients are designated by τ and μ. In order to illustrate the impacts of dispersion, we will graphically depict the results by changing the values of the parameters related to the solutions obtained in this section. It is important to note that we have constructed explicit and stable wave solutions to the Gilson-Pickering equation in a variety of ways, including using hyperbolic, trigonometric, exponential, rational, and other functions. By setting numerous individual values to the derived solutions, the soliton profile for each solution is formed. We have drawn the 3D and 2D combined line plots of the solution $|u_1(x,t)|$ of the gained wave solution. The 3D wave phenomena represent the singular periodic wave profile of the solution (2.3) and wave propagates along x axis, which is depicted in Fig. 1(A) due to the values of $\alpha = 0.001, \beta = 0.5, \gamma = 1, \kappa = 0.1$ and $\upsilon = 1.01$ within the interval $-10 \le x,t \le 10$. As a result of changing the values of $\alpha = 0.005$ and 0.009, we get same type of 3D wave structure which are shown in Fig. 1(B, C). We observed that for different soliton values of α, the amplitude component of the solution are different, is exposed in Fig. 1 (D).

We have also drawn the 3D and 2D combined line plots of the gained wave solution $|u_9(x,t)|$. The 3D wave profile depicts the kink soliton of the solution (2.7) as shown in Fig. 2 (A) due to the values $\alpha = 0.01, \beta = 0.5, \gamma = 1$ and $\upsilon = 1.01$ within the range $-10 \le x,t \le 10$. Changing the value of $\alpha = 0.03$ and 0.05, we get same type of 3D wave structure which are shown in Fig. 2(B, C). We perceive that the amplitude component of the solution differs for various parametric values of α, is exposed in Fig. 2(D).

The wave structure of the solution $|u_{33}(x,t)|$ represents a U shape wave profile for the value $\alpha = 0.9$ which is shown in Fig. 3(A). This characteristic remain unchanged after increasing the value $\alpha = 1.04$ which is shown in Fig. 3(B), and also Fig. 3(C) shows a U shape wave profile for the value $\alpha = 1.08$. In addition, illustrations of 2D combined line plots are shown in Fig. 3(D). It is seen that the wave profile of Fig. 3 (D) are similar but the amplitude and phase of the wave profile are different.

The constraint $\gamma^2 + 4\upsilon^2 > 0$, $\kappa \ne 0$ and $\kappa = -\upsilon$ is imposed in the solution $u_{41}(x,t)$. The solution of the Gilson-Pickering equation, which is a hyperbolic function, is shown in the cases of 3D and 2D waves. For the values $\alpha = 0.1, \beta = 0.2, \gamma = 0.1, \upsilon = 0.3$ inside the displacement $-10 \le x,t \le 10$, the 3D waves exhibit the bell-shaped soliton and the wave propagates along the x axis, depicted in Fig. 4(A). We achieve the same type of 3D wave structure as illustrated in Fig. 4(B, C) by adjusting the value of $\upsilon = 0.11$ and $\upsilon = 0.12$. As illustrated in Fig. 4(D), the varied values of υ, and the phase and amplitude component of the solution (2.26) are frequently different.

For $\kappa = 0.02$, the solution $u_{71}(x,t)$ corresponds to the soliton profile of the parabolic type shown in Fig. 5(A). On the other hand, as shown in Fig. 5(B) and Fig. 5(C), there was no change in the shape of the soliton profiles even after increasing the value of κ from 0.04 to 0.06. The 2D combined line plot of the solution $u_{71}(x,t)$ at $t = 0$, corresponding to the 3D wave profile for different value of κ are depicted and presented in Fig. 5(D).

Comparison

In this section, we compare the solutions obtained in this article with the analytic wave solutions accessible in the literature of the GP equation. Some approaches for studying the GP equation were used in the early literature, including the $(G'/G, 1/G)$-expansion method [36], the sinh-Gordon function method [36], the tanh-coth method [40] etc. We compare the obtained solutions to those established by Yokus et al., [36] and Bilal et al., [39] solutions. More precisely, Bilal et al., [39] considered the (G'/G^2)-expansion and expansion function methods and obtained soliton solutions of the GP equation. On the other hand, Yokus et al., [36] considered the $(G'/G, 1/G)$-expansion and the sinh-Gordon function methods and obtained soliton solutions of the GP equation.

It is noted that Bilal et al., [39] found fourteen exact solutions to the GP equation using the (G'/G^2)-expansion method and fourteen exact solutions by using expansion function method. On the other hand, the advanced auxiliary equation method is used to create lot of wave solutions of the well-known GP equation. Both methods have some common solutions as shown in Tables 1 and 2. It is observed that we have established huge amount wave solutions than the solutions determined by the (G'/G^2)-expansion method.

We also compare the solutions found by Yokus et al., [36] and the solutions established in this article. By using the sinh-Gordon function method, Yokus et al., [36] achieved only four exact solutions to the GP equation. Furthermore, the $(G'/G, 1/G)$-expansion method was also used to investigate analytical wave solutions to the GP equation and found ten solutions in [36] by Yokus et al., On the contrary, by the advanced auxiliary equation method, we attain a lot of soliton solutions of the described equation. These solutions are expressed by the rational function, hyperbolic function, exponential function and trigonometric function. Finally, we can say that we have got a huge number of wave solutions than the sinh-Gordon function and the $(G'/G, 1/G)$-expansion methods by using the suggested method.

Conclusion

We have successfully applied the advanced auxiliary equation technique to the GP equation in this article and obtained a large number of analytical wave solutions to this equation using this technique. The analytical wave solutions comprise different parameters that provide the periodic, kink-shaped, bell-shaped, U-shaped, parabolic-shaped solitons and some other shapes for specific values. We displayed the 3D and 2D combined plots of some selected solutions by changing the values of the parameters. The 2D combined plots indicate the impact of the dispersion coefficient and free parameters in the figures. The Figs. 1-5 demonstrate that the dispersion coefficient has greater impact than the free parameters in changing the nature and position of the solitons. We have also compared the results obtained with the results available in the previous literature. This study shows that the advanced auxiliary equation method appears to be robust, compatible and simple to use, and it provides lots of wave solutions of NLPDEs with various free parameters that are useful to describe the wave profile in science and engineering. The method could be used to investigate a variety of additional NLPDEs and it is our future endeavor.

Data availability

This manuscript has no associated data.

CRediT authorship contribution statement

S.M. Rayhanul Islam: Conceptualization, Formal analysis, Writing – original draft, Investigation, Resources, Validation. **Shahansha Khan:** Software, Validation, Writing – review & editing. **S.M. Yiasir Arafat:** Data curation, Methodology, Visualization. **M. Ali Akbar:** Supervision, Project administration, Funding acquisition.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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