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Research article

Multi-soliton, breathers, lumps and interaction solution to the (2+1)-dimensional asymmetric Nizhnik-Novikov-Veselov equation

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ABSTRACT

In this work, we consider a (2 + 1)-dimensional asymmetric Nizhnik-Novikov-Veselov (ANNV) equation, which has applications in processes of interaction of exponentially localized structures. Based on the bilinear formalism and with the aid of symbolic computation, we determine multi-solitons, breather solutions, lump soliton, lump-kink waves and multi lumps using various ansatze's function. We notice that multi-lumps in the form of breathers visualize as a straight line. To realize dynamics, we commit diverse graphical analysis on the presented solutions. Obtained solutions are reliable in the mathematical physics and engineering.

1. Introduction

Nonlinear phenomena have an extensive application in different branches of mathematical physics and engineering. The explicit solutions of NLEEs play a prominent role in the study of nonlinear science. Various effective procedure have been developed to solve NLEEs, like the inverse scattering transform [1], the Darboux transformation [2], Backlund transformation [3], the unified method (UM) and its generalized form (GUM) [4, 5] and Hirota bilinear form method [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25]. The Hirota's bilinear method is one of the most direct and convenient method to obtain the exact soliton solution of NLEEs. If a NLEE can attain its bilinear form, Lax pairs, lump solutions, multiple soliton solutions of this equation can be obtained.

Recently, researchers are highly impressed to rogue wave solutions [9, 10] for it's engrossing class of lump-type solutions, which can be found in plasma, shallow-water waves, nonlinear optics and Bose-Einstein condensates [11]. In 2002, Lou et al. studied the lump solution with the variable separation method [12]. Very recently, Ma et al. proposed the positive quadratic function to get the lump solution. Special examples of lump type solutions have been found, such as the KPI equation [13], Boussinesq equation [14], BKP equation [15] and so on. Lump solution [16, 17] is a kind of rational function solution which is localized in all directions in the space whereas lump-type [18, 19] solutions are localized in almost all directions in the space. Rogue waves [19, 20, 21, 22] are localized in both space and time, arise from nowhere and disappear without a trace [23], have taken the responsibility for unexpected disaster in the world.

In this paper, we will examine the (2 + 1)-dimensional asymmetric Nizhnik-Novikov-Veselov (ANNV) equation [3, 25],

$$u_t + u_{xxx} + 3[uv]_x = 0; \quad u_x = v_y \tag{1}$$

where u and v are the components of the (dimensionless) velocity [26]. Eq. (1) is the only known isotropic Lax extension of the Korteweg-de Vries equation [27]. The ANNV equation has important applications in incompressible fluids, such as shallow-water waves, long internal waves and acoustic waves. There are many researchers have been studied in ANNV equation in many ways such as: Boiti et al. solved via the inverse scattering transformation [28]. Guo et al. discussed the N-soliton solution and Pfaffian expression by using a nonlinearized method of Lax pair [3], Osman et al. solved this system of equations via the unified and generalized unified method [29, 30, 31, 32]. Also, ANNV equations can also be obtained from the inner parameter-dependent symmetry constraint of the KP equation [33].

The main purpose in this work is to present the direct approach to construct some solutions such as multi-soliton solutions, lump solution and lump-kink wave solutions for ANNV equation. Also, we have discussed some new phenomena such as breather wave from two soliton

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solution, non-elastic solution and multi-lump wave solutions for ANNV equation.

1.1. Bilinear form

Let us introduce the following potential transformation

$$u = c(t)q_{xy} \quad \text{and} \quad v = c(t)q_{xx} \tag{2}$$

in which $c = c(t)$ is a function to be known later. Substituting (2) into (1) and integrating the equation with respect to x once and taking $c = 1$, we get

$$E(q) = q_{yt} + q_{xxxy} = 0. \tag{3}$$

by choosing the integration constant as zero. Based on the results presented in Refs. [34, 35], we obtain

$$E(q) = P_{yt}(q) + P_{xxxy}(q) = 0. \tag{4}$$

with the help of the following two important transformations, we get

$$\begin{cases} q = 2\ln f(x,y,t) \Leftrightarrow u = cq_{xy} = 2[\ln f(x,y,t)]_{xy} \\ q = 2\ln f(x,y,t) \Leftrightarrow v = cq_{xx} = 2[\ln f(x,y,t)]_{xx} \end{cases}. \tag{5}$$

Substituting above transformations (5) into Eq. (1), (2 + 1)-dimensional asymmetric Nizhnik-Novikov-Veselov equation can be linearized into

$$(D_yD_t + D_yD_x^3)f \cdot f = 0, \tag{6}$$

2. Results & discussion

2.1. The 1-soliton solution

To seek one-soliton solutions of Eq. (1), we suppose f is expressed in the following form

$$f = a_0 + a_1 e^{l_1x+m_1y+n_1t} \tag{7}$$

where $l_1, m_1, n_1,\ a_i, (i = 0, 1)$ are arbitrary constants to be determined later. Inserting Eq. (7) into Eq. (6) and after some simplification, equating all the coefficient of exponential term to be zero, we can obtain the set of algebraic equations for $l_1, m_1, n_1,\ a_i, (i = 0, 1)$. Solving the system with the aid of symbolic computation system Maple, we obtain the subsequent solution:

$$a_0 = a_0, a_1 = a_1, l_1 = l_1, m_1 = m_1, n_1 = -l_1^3. \tag{8}$$

Therefore, substituting Eqs. (7) and (8) along with Eq. (5) into Eq. (6), yields the desired one-soliton solution of Eq. (1), which is exposed in Fig. 1.

2.2. The 2-soliton solution

To seek two-soliton solutions of Eq. (1), we choose f is expressed as

$$f = a_0 + a_1 e^{l_1x+m_1y+n_1t} + a_2 e^{l_2x+m_2y+n_2t} + a_3 e^{l_1x+m_1y+n_1t+l_2x+m_2y+n_2t} \tag{9}$$

where $a_i, (i = 0, 1)$, $l_i, m_i, n_i (i = 1, 2)$ are all real parameters to be determined. Substituting Eq. (9) into Eq. (6) and after some simplification, equating all the coefficient of exponential term to be zero, we can obtain

$$\left\{ a_0 = \frac{a_1a_2(l_1 - l_2)(m_1 - m_2)}{a_3(l_1 + l_2)(m_1 + m_2)}, a_i = a_i, (1 \le i \le 3), l_1 = l_2 = m_1 = m_2 = const., n_1 = -l_1^3, n_2 = -l_2^3. \right. \tag{10}$$

which should satisfies the conditions $a_3 \neq 0$, $(l_1 + l_2) \neq 0$, and $(m_1 + m_2) \neq 0$.

Therefore, substituting Eqs. (9) and (10) along with Eq. (5) into Eq. (6), yields the desired two- soliton solution. If we setting $a_1 = 1,\ a_2 = 1,\ a_3 = 10, l_1 = 2, l_2 = 2.5,\ m_1 = 1, m_2 = 3.5$, we can obtain a two-soliton solution of Eq. (1). If we setting $l_1 \neq -l_2 \in \Re$ and $m_1 = m_2 \in \Re$, then we obtain another type of two soliton solution. First type solution is elastic but second type is non-elastic solution, which are exposed in Figs. 2 and 3 respectively.

Based on the above method Eq. (9) gives the breathers by asset of selecting suitable parameters. Breather solutions of Eq. (1) can be obtained in the (x, y) plane, where the parameters in Eq. (10) meeting the following conditions

$$l_1 = Ib_1, l_2 = -Ib_2, a_1 = k_1, a_2 = k_2, a_3 = k_3, m_1 = b + Ik, m_2 = b - Ik. \tag{11}$$

For instance, setting parameters as follows $l_1 = I,\ l_2 = -2I,\ m_1 =$

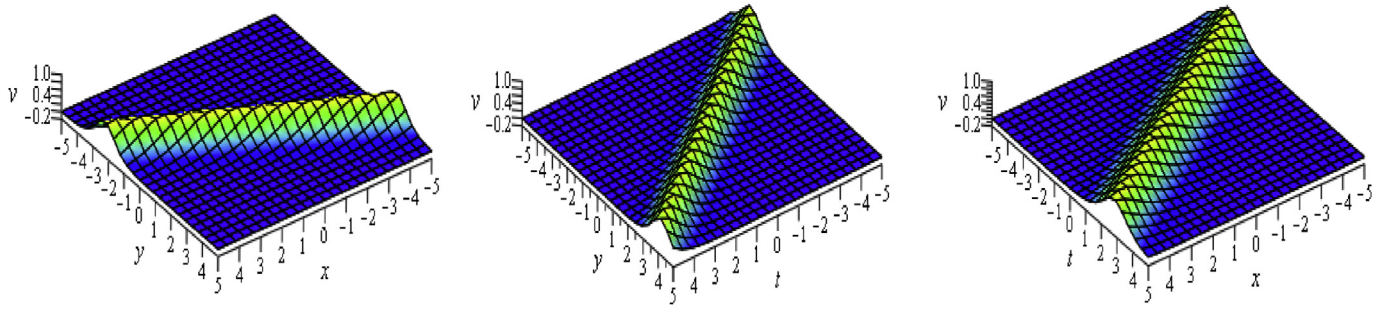


Fig. 1. The one-stripe soliton solution for Eq. (1) by choosing suitable parameters:$a_0 = 2, a_1 = 1, l_1 = 1.25, m_1 = 2.5$, 3D shape in different planes at (a) $t = 0$; (b) $x = 0$; and (c) $y = 0$.

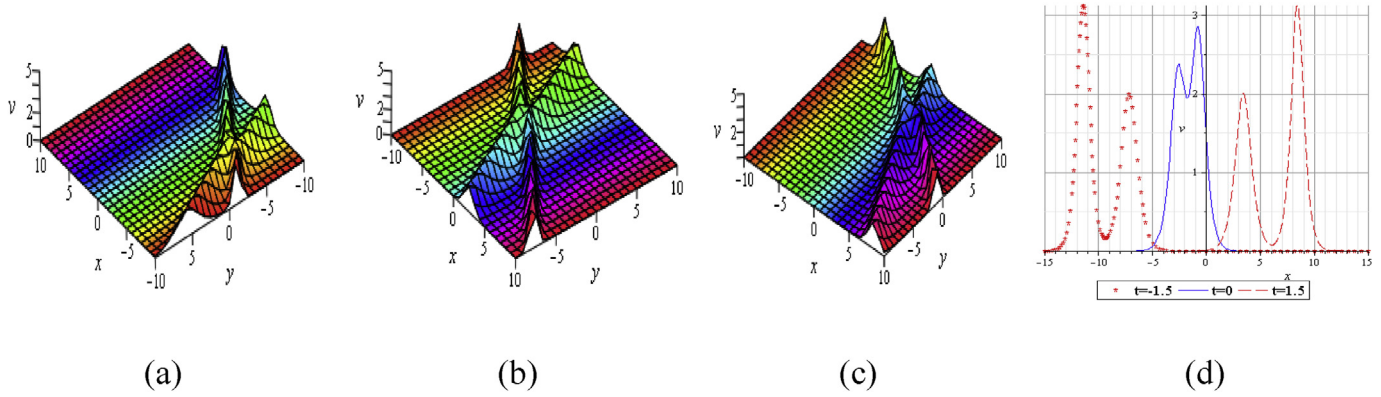


Fig. 2. The two-stripe soliton solution for Eq. (1) by choosing suitable parameters: $a_1 = 1,\ a_2 = 1, a_3 = 10,\ l_1 = 2, l_2 = 2.5,\ m_1 = 1, m_2 = 3.5$, with 3D plots for different times (a) $t = -1.5$; (b)$t = 0$; and (c) $t = 1.5$ respectively, (d) Corresponding 2D plot.

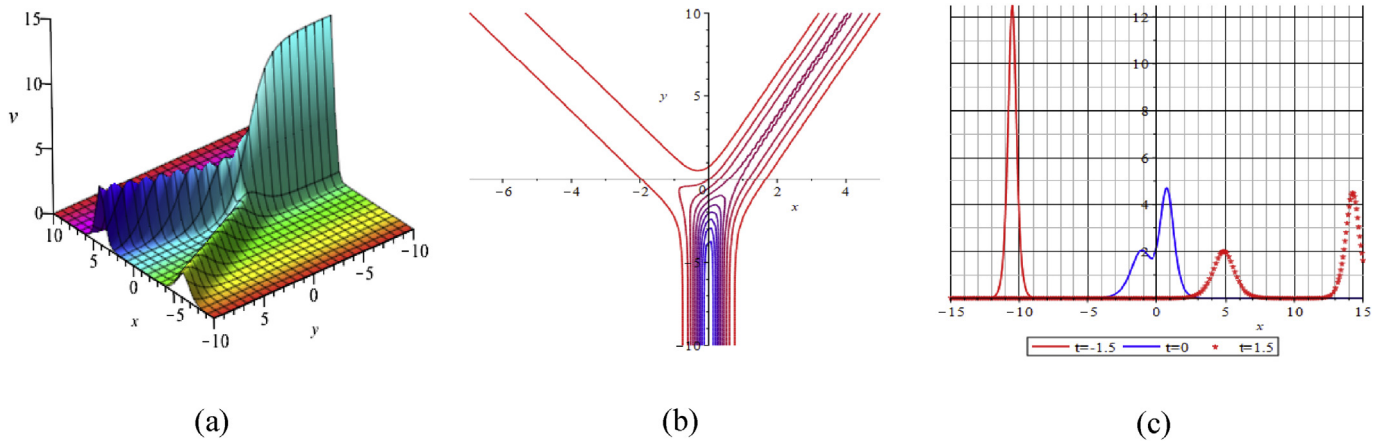


Fig. 3. The two-stripe soliton solution (non-elastic) for Eq. (1) by choosing suitable parameters: $a_1 = 1,\ a_2 = 1, a_3 = 10, l_1 = 2, l_2 = -3,\ m_1 = m_2 = 1$, at time $t = 0$ (a) 3D plot (b) Contour plot and (c) Corresponding 2D plot for different time.

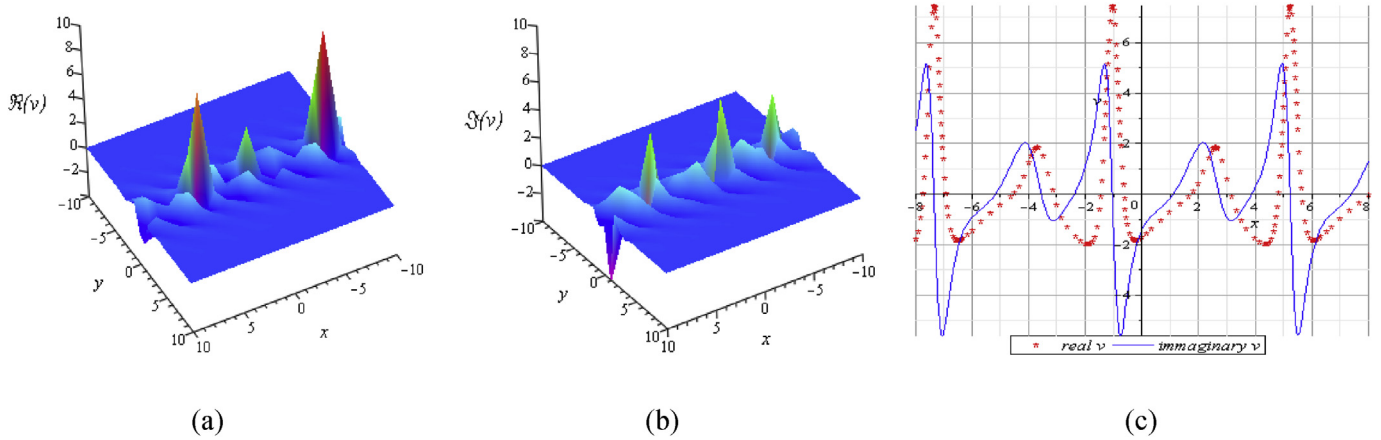


Fig. 4. The breather solution for Eq. (1) by choosing suitable parameters: $a_1 = 1, a_2 = 1, a_3 = 10, l_1 = 2, l_2 = 2.5,\ m_1 = 1, m_2 = 3.5$, with $t = 0$: 3D plots (a), (b) and (c) Corresponding 2D plot.

$m_2^* = 1 + 2I,\ a_1 = 1.25,\ a_2 = 1.5,\ a_3 = 2$, we can obtain breathers and gives the wave shape at $t = 0$, which is shown in Fig. 4.

During the wave propagation, we see that the amplitude, velocity and envelop shape of the one-soliton keep constant (see Fig. 1). One can confirm that the amplitudes of impatient position are limited and around same in different spaces.

As depicted from Fig. 2, the collision is elastic between two bell-shaped solitons, because the velocities, amplitudes and envelop shapes of a moving soliton always keep fixed their shapes after the interaction. All the phenomena concludes that energy will remain unchanged during collision. Whereas we see that from Fig. 3, the interaction between two bell-shaped solitons is completely non-elastic. That is the soliton velocity, amplitude and wave shape are changed after collision.

Now we will illustrate the wave pattern situations of solitary wave by

three figures. Fig. 1 highlight the one-soliton (7), Figs. 2 and 3 demonstrates the two-soliton solution (9) and Fig. 4, special type solution of Eq. (9) called breather solution, by choosing suitable parameters.

2.3. The 3-soliton solution

To seek three-soliton solutions of Eq. (1), we suppose f is expressed as

$$f = a_0 + e^{\phi_1} + e^{\phi_2} + e^{\phi_3} + a_{12}e^{\phi_1+\phi_2} + a_{23}e^{\phi_2+\phi_3} + a_{13}e^{\phi_1+\phi_3} + a_{123}e^{\phi_1+\phi_2+\phi_3} \tag{12}$$

with

$$\phi_i = l_i x + m_i y + n_i t, i = 1, 2, 3 \tag{13}$$

where $a_0, a_{12}, a_{23}, a_{13}, a_{123}, l_i, m_i, n_i (i = 1, 2, 3)$ are all real parameters to be determined. Based on above method, substituting Eq. (12) with Eq. (13) into Eq. (6), we can obtain the following relations among parameters

$$\begin{cases} a_0 = \dfrac{(l_1 - l_2)(2l_3 - l_2)}{a_{13} l_2 (l_1 - l_2 + 2l_3)}, m_1 = a_{12} = a_{23} = a_{123} = 0, a_{13} = l_1 = l_2 = l_3 = m_2 = m_3 = const., \\ n_1 = -3l_1^2 l_3 - 3l_1 l_3^2 + 3l_1^2 l_2 - 3l_1 l_2^2 - l_1^3 + 6l_1 l_2 l_3, n_2 = -l_2^3 + 3l_2^2 l_3 - 3l_2 l_3^2, n_3 = -l_3^3. \end{cases} \tag{14}$$

which needs to satisfy the condition $a_{13}, l_2 \neq 0$.

Therefore, substituting Eqs. (12), (13) and (14) along with Eq. (5) into Eq. (6), the three-soliton solution of Eq. (1) can be obtained, which is shown in Fig. 5.

2.4. Lump solutions of the (2+1)-dimensional ANNV equation

To seek lump solutions of Eq. (1), we suppose f is expressed in the following form:

$$f = g^2 + h^2 + p_1, \tag{15}$$

where,

$$g(x, y, t) = l_1 x + m_1 y + n_1 t, \quad \text{and} \quad h(x, y, t) = l_2 x + m_2 y + n_2 t, \tag{16}$$

where $p_1, l_i, m_i, n_i (i = 1, 2)$ are all real constants to be determined. A direct symbolic computation with f gives rises to the following relations:

$$p_1 = p_1, l_1 = -\frac{m_2 l_2}{m_1}, l_2 = l_2, m_1 = m_1, m_2 = m_2, n_1 = 0, n_2 = 0. \tag{17}$$

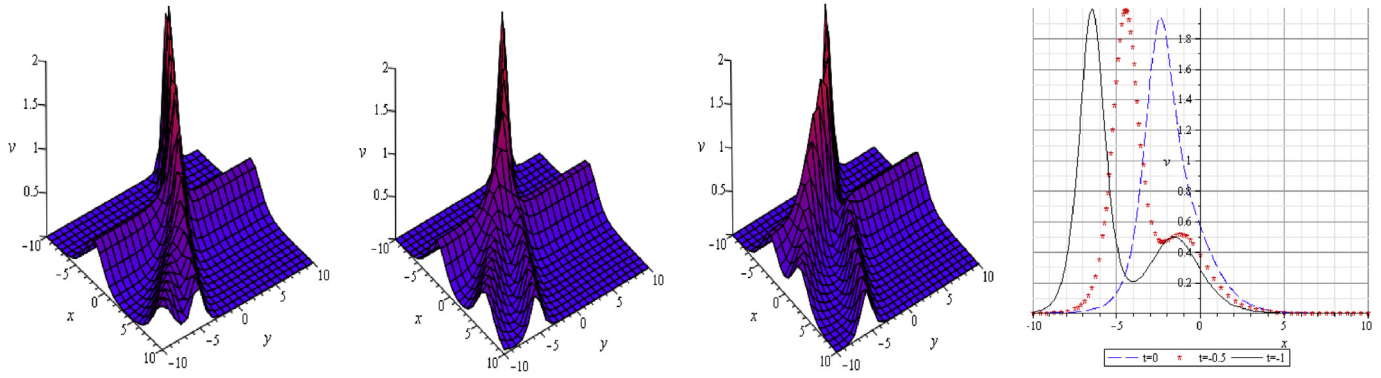
Therefore, substituting Eq. (17) with Eq. (16) into Eq. (15), we can get a class of quadratic function solutions Eq. (5). Then, the resulting exact rational solution for Eq. (1) are obtained through the transformation

$$u = 2(\ln f)_{xy} = \frac{4(l_1 m_1 + l_2 m_2)p_1 - 8gh(l_1 m_2 + l_2 m_1) + 4(l_1 m_1 - l_2 m_2)\left(-g^2 + h^2\right)}{\left(g^2 + h^2 + p_1\right)^2}, \tag{18}$$

and

$$v = 2(\ln f)_{xx} = \frac{4\left(l_1^2 + l_2^2\right)p_1 - 16 l_1 l_2 gh + 4\left(l_1^2 - l_2^2\right)\left(-g^2 + h^2\right)}{\left(g^2 + h^2 + p_1\right)^2}, \tag{19}$$

where $g(x, y, t) = l_1 x + m_1 y + n_1 t$, $h(x, y, t) = l_2 x + m_2 y + n_2 t$, for example, the resulting solutions of Eq. (17) are as follows



(a) (b) (c) (d)

Fig. 5. The three-stripe soliton solution for Eq. (1) by choosing parameters: $a_{13} = 2, l_1 = 1, l_2 = 2, l_3 = 2, m_2 = 1, m_3 = 3$, with 3D plots at (a) $t = 0$, (b) $t = -0.5$, and (c) $t = -1$ respectively, (d) 2D plot at $t = 0, -0.5$ and $t = -1$ respectively.

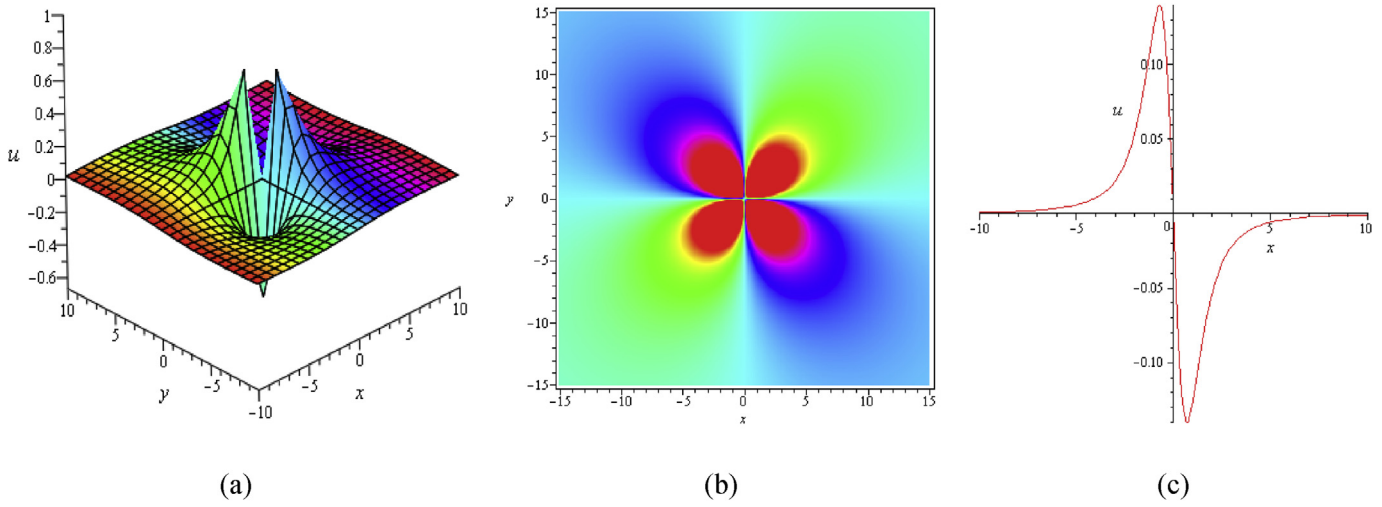


Fig. 6. Lump solution u in (20) Eq. (1) by choosing suitable parameters: $p_1 = 1.2, l_2 = 0.8,\ m_1 = -0.8$, and $m_2 = 0.4$ (a) 3-D plot of u (b) density plot of u (c) 2-D plot of u.

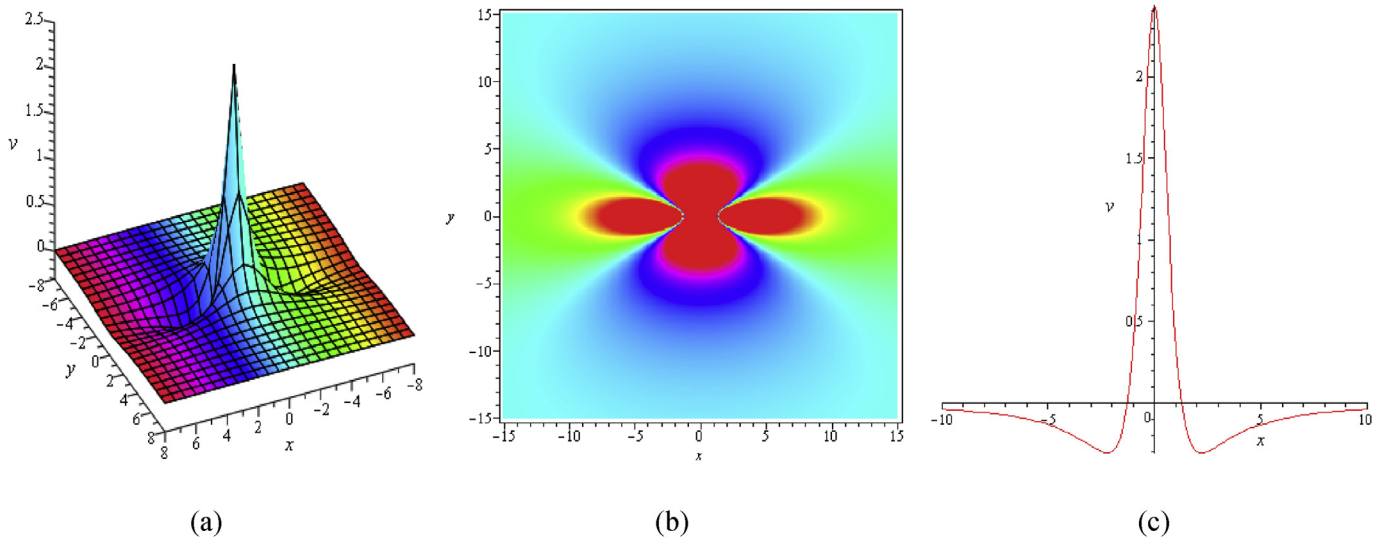


Fig. 7. Lump solution v in (20) for Eq. (1) by choosing suitable parameters:$p_1 = 2, l_2 = 1,\ m_1 = 2$, and $m_2 = 1$. (a) 3-D plot of v (b) density plot of v (c) 2-D plot of v.

$$u = -\frac{8\left(-\frac{gm_2l_2}{m_1} + hl_2\right)(gm_1 + hm_2)}{\left(g^2 + h^2 + p_1\right)^2},\quad v = \frac{4\left(\frac{m_2^2l_2^2}{m_1^2} + l_2^2\right)}{g^2 + h^2 + p_1} - \frac{4\left(-\frac{gm_2l_2}{m_1} + hl_2\right)^2}{\left(g^2 + h^2 + p_1\right)^2}, \tag{20}$$

with the function g and h are given as follows

$$g = -\frac{m_2l_2x}{m_1} + m_1y,\quad \text{and}\quad h = l_2x + m_2y. \tag{21}$$

For the exact solution $u(x, y, t)$ and $v(x, y, t)$ to Eq. (1) to be lump ones, it is observed that

$$\lim_{x^2+y^2\to\infty} u(x, y, t) = 0,\quad \text{and}\quad \lim_{x^2+y^2\to\infty} v(x, y, t) = 0,\quad \forall t \in \Re. \tag{22}$$

It is easy to see that for any given time t, the lump solutions $u \to 0, v \to 0$, if and only if the corresponding summation of squares $g^2 + h^2 \to \infty$, which is equivalent to $x^2 + y^2 \to \infty$.

Substituting the noted values of $p_1, l_2, m_i (i = 1, 2)$ into Eq. (20), then we can get abundant exact lump solutions of Eq. (1). We can notice that the solutions we obtained have a unified form of (19). If we taking the values of $t = t_0$, then the coordinates of the central point of the obtained lump solution is

$$\left(x = \frac{m_1n_2t_0 - m_2n_1t_0}{l_1m_2 - l_2m_1}, y = \frac{n_2l_1t_0 - n_1l_2t_0}{l_1m_2 - l_2m_1}\right) \tag{23}$$

where $l_1m_2 - l_2m_1 \neq 0$. Substituting Eq. (23) and $t = t_0$ into Eq. (19), the amplitude of v is attained $\text{Max}(v) = \frac{4(l_1^2+l_2^2)}{p_1}(p_1 \neq 0)$, from which we observe that the amplitude of the lump solution is depend on the values of l_1, l_2 and p_1. As we seen from Eq. (23) the lump soliton is centered at the origin when $t = 0$.

Fig. 6 shows the sketch the lump solution u in Eq. (20) whereas Fig. 7 shows the sketch lump of v in Eq. (20) called rogue waves for some values $p_1 = 2, l_2 = 1,\ m_1 = 2$, and $m_2 = 1$, (a) gives 3D views from which can expose the standard rogue wave features. It is also clear that the Fig. 7a is

the well-known eye-shaped rogue wave solution which has two valleys and one local hump. Moreover, we notice that rogue wave has the highest peak in its surrounding waves and forms in a tiny time, which is clear from Fig. 7c. For fixed t, the variables can determine the rogue wave is symmetric about the x axis (see Fig. 7b).

2.5. Interaction of lump waves with solitary waves

To get the interaction phenomena between lumps and solitary waves solutions of Eq. (1), assuming $f(x,y,t)$ in the following new form.

$$f = g^2 + h^2 + p_1 + \lambda \exp(\eta), \tag{24}$$

with

$$g(x,y,t) = l_1x + m_1y + n_1t, h(x,y,t) = l_2x + m_2y + n_2t \text{ and } \eta(x,y,t) = l_3x + m_3y + n_3t, \tag{25}$$

where $p_1, l_i, m_i, n_i (1 \le i \le 3)$ are all real parameters to be determined. Substituting Eq. (24) along with Eq. (25) into Eq. (6) with the aid of symbolic computation system Maple, we can obtain the following relations among parameters:

$$l_2 = -\frac{l_1m_1}{m_2}, n_3 = -l_3^3, m_3 = n_1 = n_2 = 0, p_1 = \lambda = l_1 = l_3 = m_1 = m_2 = const. \tag{26}$$

which should satisfy $m_2 \neq 0$.

Therefore, substituting Eq. (26) into Eq. (24), we can get a class of quadratic function solutions to the bilinear Eq. (6). Then, the resulting exact rational solution for Eq. (1) are obtained through the transformation,

$$u = 2(\ln f)_{xy} = \frac{\begin{array}{c}4(l_1m_1 + l_2m_2)p_1 - 8gh(l_1m_2 + l_2m_1) + 4(l_1m_1 - l_2m_2)\left(-g^2 + h^2\right) + \\ 2\left\{\left(g^2 + h^2 + p_1\right)l_3m_3 + (l_1m_1 + l_2m_2) - (l_1m_3 + l_3m_1)g - (l_2m_3 + l_3m_2)h\right\}\lambda e^{\eta}\end{array}}{\left(g^2 + h^2 + p_1\right)^2}, \tag{27}$$

$$v = 2(\ln f)_{xx} = \frac{4\left(l_1^2 + l_2^2\right)p_1 - 16ghl_1l_2 + 4\left(l_1^2 - l_2^2\right)\left(-g^2 + h^2\right) + 2\left\{\begin{array}{c}\left(g^2 + h^2 + p_1\right)l_3^2 + \\ 2\left(l_1^2 + l_2^2\right) - 4(gl_1 + hl_2)l_3\end{array}\right\}\lambda e^{\eta}}{\left(g^2 + h^2 + p_1\right)^2}, \tag{28}$$

Where g, h and η are defined in Eq. (25).

For example, the resulting solutions of Eq. (26) are as follows

$$u = -\frac{8\left(gl_1 - \frac{hl_1m_1}{m_2} + \frac{1}{2}\lambda l_3e^{\eta}\right)(gm_1 + hm_2)}{\left(g^2 + h^2 + p_1 + \lambda e^{\eta}\right)^2}, v = \frac{4\left(l_1^2 + \frac{l_1^2m_1^2}{m_2^2} + \frac{1}{2}\lambda l_3^2e^{\eta}\right)}{g^2 + h^2 + p_1 + \lambda e^{\eta}} - \frac{4\left(gl_1 - \frac{hl_1m_1}{m_2} + \frac{1}{2}\lambda l_3e^{\eta}\right)}{\left(g^2 + h^2 + p_1 + \lambda e^{\eta}\right)^2}. \tag{29}$$

where $g = l_1x + m_1y$, $h = -\frac{l_1m_1}{m_2}x + m_2y$ and $\eta = l_3x - l_3^3t$. (30)

In what follows, Fig. 8 presents exact solution of Eq. (29) by choosing the suitable parameters, which can show the interaction phenomena between solitary wave and lump waves.

2.6. Multi lump solutions of (2 + 1)-dimensional ANNV equation

In this section, we will find the multi lump solution of Eq. (1). To this aim, the above function $f(x,y,t)$ can be taken as,

$$f = e^{-\psi_1} + h_1e^{\psi_1} + h_2\sin\psi_2, \tag{31}$$

with $\psi_1 = p_1(x + n_1y - w_1t)$ and $\psi_2 = p_2(x + n_2y - w_2t)$, (32)

where $p_i, n_i, w_i (i = 1, 2)$ are all real parameters to be determined.

Substituting Eq. (31) along with Eq. (32) into Eq. (6) with the aid of symbolic computation system Maple, we can obtain the following relations among parameters

which should satisfy $h_1, p_1 \neq 0$.

Under the transformation Eq. (5), we can get the periodic lump solutions of the (2 + 1)-dimensional ANNV equation as,

$$n_1 = -\frac{1}{4}\frac{h_2^2 p_2^2 n_2}{h_1 p_1^2}, w_1 = p_1^2 - 3p_2^2, w_2 = -p_2^2 + 3p_1^2, h_1 = h_2 = n_2 = p_1 = p_2 = const., \tag{33}$$

$$u = \frac{2\left(-p_1\zeta_1 + p_1\zeta_2 - h_2 p_2^2 n_2 \sin\delta_2\right)}{\left(e^{-\delta_1} + h_1 e^{\delta_1} + h_2 \sin\delta_2\right)} - \frac{2(\zeta_1 + \zeta_2 + h_2 p_2 n_2 \cos\delta_2)\begin{pmatrix} -p_1 e^{-\delta_1} + h_1 p_1 e^{\delta_1} \\ +h_2 p_2 \cos\delta_2 \end{pmatrix}}{\left(e^{-\delta_1} + h_1 e^{\delta_1} + h_2 \sin\delta_2\right)^2}, \tag{34}$$

and

$$v = \frac{2\left(p_1^2 e^{-\delta_1} + h_1 p_1^2 e^{\delta_1} - h_2 p_2^2 \sin\delta_2\right)}{\left(e^{-\delta_1} + h_1 e^{\delta_1} + h_2 \sin\delta_2\right)} - \frac{2\left(-p_1 e^{-\delta_1} + h_1 p_1 e^{\delta_1} + h_2 p_2 \cos\delta_2\right)^2}{\left(e^{-\delta_1} + h_1 e^{\delta_1} + h_2 \sin\delta_2\right)^2}, \tag{35}$$

where

$$\begin{cases} \zeta_1 = \frac{h_2^2 p_2^2 n_2 e^{-\delta_1}}{4 h_1 p_1}, \zeta_2 = -\frac{h_2^2 p_2^2 n_2 e^{\delta_1}}{4 p_1}, \delta_1 = p_1\left(x - \frac{1}{4}\frac{h_2^2 p_2^2 n_2 y}{h_1 p_1^2} - \left(p_1^2 - 3p_2^2\right)t\right) \\ \delta_2 = p_2\left(x + n_2 y - \left(-p_2^2 + 3p_1^2\right)t\right). \end{cases} \tag{36}$$

In what follows, Figs. 9 and 10 present exact solution of Eq. (34) and Eq. (35) respectively by choosing the suitable parameters, which can demonstrate the interaction phenomena among multi lump solution.

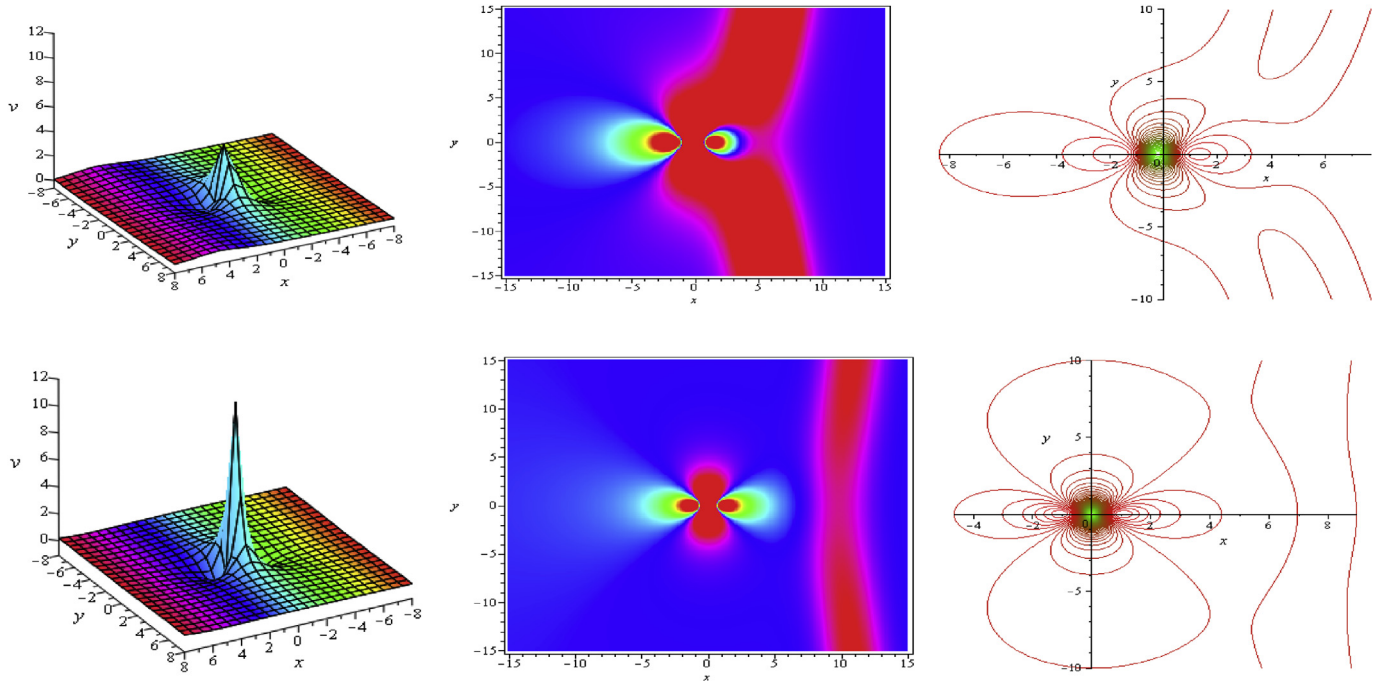


Fig. 8. Profiles of v in (29) with $t = 0$: 3d plots, density plot and contour plot (top for $a_2 = 5$) and bottom for $a_2 = 0.05$ by choosing suitable parameters: $a_1 = 2.5$, $m_1 = 2.3$, $m_2 = 1$, $l_1 = 1.5$, and $l_3 = 1$.

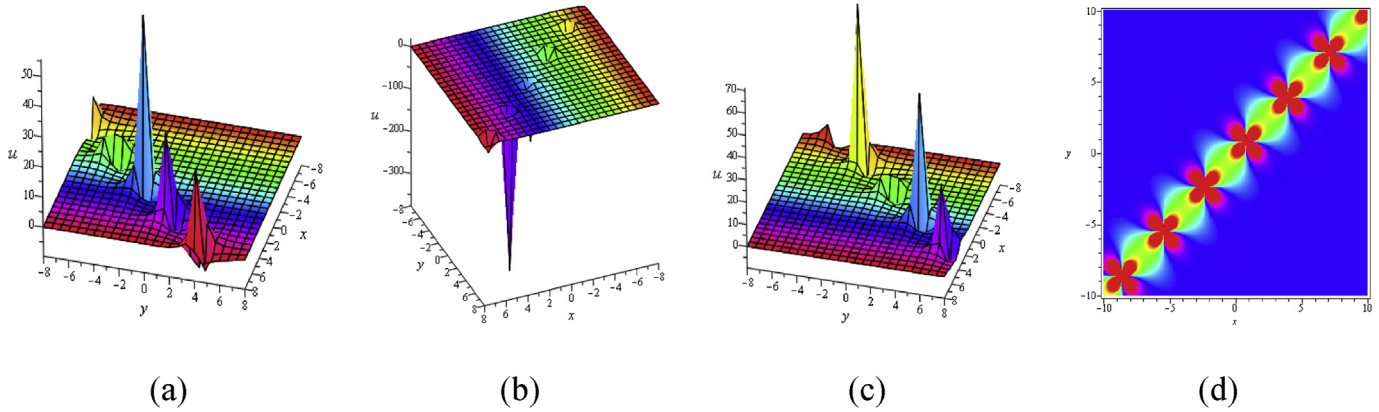


Fig. 9. Profiles of u in (34) with $t = -1.5, 0, 1.5$: 3d plots (a), (b), (c) respectively and (d) correspon-ding density plot (b) by choosing suitable parameters: $h_1 = 1$, $h_2 = 2$, $p_1 = 1, p_2 = -1$ and $n_2 = 1$.

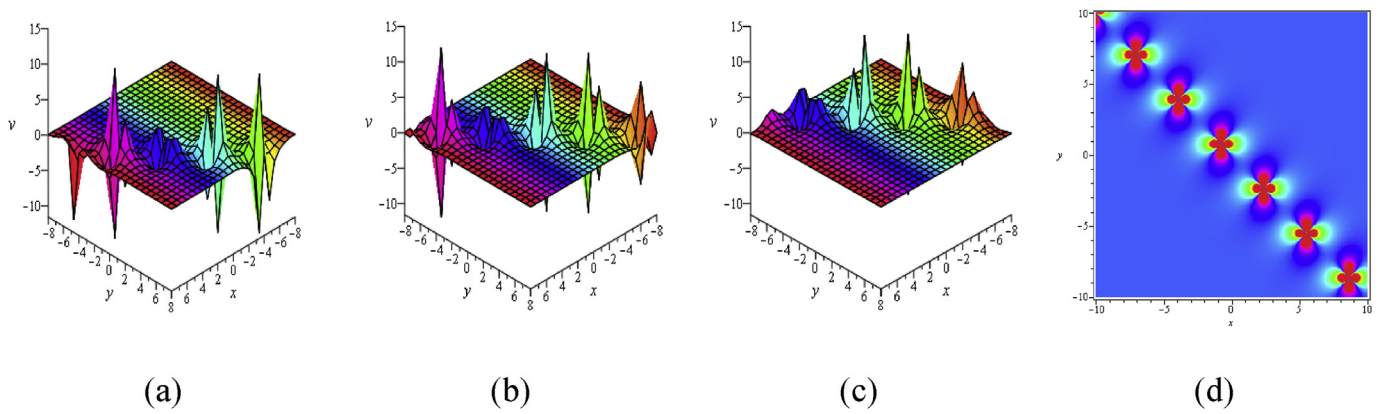


Fig. 10. Profiles of v in (35) with $t = -2, 0, 2$: 3d plots (a), (b), (c) respectively and (d) corresponding density plot (b) by choosing suitable parameters:$h_1 = 1, h_2 = 2, p_1 = 1, p_2 = -1$ and $n_2 = -1$.

3. Conclusion

In conclusion, the (2 + 1)-dimensional asymmetrical NNV equations has been investigated. We have derived soliton solution, breathers, lump solutions, mixed lump stripe solutions based on bilinear method and symbolic computation. Some obtained results are shown graphically in order to demonstrate that the technique is quite efficient for handling nonlinear equations. Meanwhile, the performances of the mentioned techniques are substantially powerful and absolutely reliable to search new explicit solutions of other NPDEs.

Declarations

Author contribution statement

M. Belal Hossen: Conceived and designed the analysis; Wrote the paper.

Harun-Or-Roshid: Analyzed and interpreted the data.

M. Zulfikar Ali: Contributed analysis tools or data.

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Additional information

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