Hindawi
Mathematical Problems in Engineering
Volume 2023, Article ID 1647844, 19 pages
https://doi.org/10.1155/2023/1647844

Research Article

Modeling and Analysis of Stochastic Perishable Inventory System with Impatient Customers at Service Facility

Md. Amirul Islam[1,2] **Mohammad Ekramol Islam**[3] **and Abdur Rashid**[2]

[1]*Department of Mathematics, Uttara University, Dhaka, Bangladesh*
[2]*Department of Mathematics, Jahangirnagar University, Savar, Bangladesh*
[3]*Northern University Bangladesh, Dhaka, Bangladesh*

Correspondence should be addressed to Md. Amirul Islam; amirul.math@gmail.com

Received 6 June 2022; Revised 11 November 2022; Accepted 31 December 2022; Published 16 May 2023

Academic Editor: Zhihong Yao

In this paper, we consider a two-commodity stochastic (s, S) perishable inventory system of the multiqueue Jackson network at a service facility with reneging and jockeying of the customers. The waiting room capacity of the first queue is $M/M/1/\infty$ and the second queue is $M/M/1/C_2$, where $C_2 < \infty$. In this network, different queues have different service rates. Service times are exponentially distributed. Customers join the system at a rate of λ_i; $(i = 1, 2)$ using a Poisson process. We assumed that some customers maybe impatient due to the long waiting time in the queue and can leave the system without service or may switch from the longest queue to the shortest queue to reduce the waiting time. If the inventory levels of the items from the warehouse i at any time reaches below or reorder level s_i, an order $Q_i = ((S_i - s_i) > 0; i = 1, 2)$ units is placed to bring the inventory level up to S_i. The joint probability distribution of the number of customers in the system and also the stock level of the warehouse is determined in the equilibrium case. Several important measures of steady-state system performance are derived. Several examples are presented to demonstrate the reneging and jockeying behavior of the customers for this system. A sensitivity analysis is performed to investigate the effect of changing the parameters on the total expected cost of the system.

1. Introduction

The queueing inventory system (QIS) with reneging and jockeying are unavoidable aspects of modern life. Jackson network of a multi $M/M/1$ type Markovian queue with attached inventory is used very extensively as a model of inventory management, supply chain management, telecommunication systems, production and manufacturing engineering and biological systems, health care systems, safety and security departments, and others. Queueing inventory models with reneging and jockeying customers have also been used to analyze service systems. Quality assurance is a more challenging issue in the service sector and the manufacturing sector of the economy. In this paper, we have designed and developed a queueing inventory model for a Jackson network to provide efficient services to customers. In making successful decisions, models play an important role in operations management. A model is a simplified version of the system that is an abstraction of reality. The main objective of service organizations is customer satisfaction. The number of customers waiting for service at a particular location in daily life. A queue is formed when customers need service facilities. In this case, the impatience behavior of the customers may affect the system of real problems. Most of the queuing models assume that patient customers are always waiting in line to get service. But some impatient customers may leave the queue in anticipation of a longer waiting time. This kind of behavior by customers without receiving the service is known as reneging. If there is more than one parallel queue in the system, customers may switch from one long queue to another short queue to reduce the waiting time. This type of behavior by customers receiving service is known as jockeying. Reneging and jockeying are common phenomena in the fields of business and industry, medical service centers, and communication networks. In such a situation, the Jackson network model was subsequently developed for providing adequate service to the customers with tolerable waiting. Since a long waiting line in the queue may cause a loss of customers when a short wait is expected.

In this research, we built a stochastic model of a perishable inventory system with impatient customers on the Jackson network that will provide a control policy to optimize the expected total cost of the system. Zhang et al. [1] developed a queueing-inventory system (QIS) model with servers' vacations and impatient customers. Krishnamoorthy et al. [2] performed a survey on the inventory system with a positive service time. Jeganathan et al. [3] demonstrated a random inventory system of (s, S) in a service facility consisting of two parallel queues of jockey and impatient customers, where the capacity of each queue has a finite size. Kumar and Sharma [4, 5], Choudhury and Medhi [6], Abou-El-Ata and Hariri [7], Montazer-Haghighi et al. [8], all discussed some multiserver Markovian queuing models with balking and reneging customers. Ekramol et al. [9–11] analyzed stochastic perishable inventory systems on the open Jackson network at a service facility. Schwarz et al. [12] analyzed product form models for queueing networks with an attached inventory. In his paper [13] Jackson analyzed queuing networks where average arrival rates across different departments were precisely determined. Krishnamoorthy et al. [14–18] analyzed the queue inventory system at service facilities with service times. Nair et al. [19] analyzed a multiserver Markovian queuing model where each server provides services to only one client. Yue et al. [20] analyzed a $M/M/1$ queuing system, in which customers lost their patience due to absenteeism. They obtain numerical results to demonstrate the influence of certain parameters on certain measures of system performance.

In the existing literature, the authors consider that the queuing inventory system (QIS) had a single service facility with one or more servers for the customers. In general, many communication systems in real life need to be represented as a network of interconnected queues for multiservice facilities for customers. A network of queues is a group of stations where customers are served at some or all of the locations. This is addressed in this paper by considering the Jackson network. It is a more reliable model than the models available in the existing literature. Customers sometimes need several services from different kinds of service centers, and when they become impatient, they leave the system or switch to another queue. For example, the service facilities in the healthcare system can be provided using the queueing network. It has a direct impact on the improvement in service quality. A patient in need of hospital treatment for multiple diagnoses from the cardiology, neurology, radiology, and gastroenterology departments, as well as from other departments of the healthcare service center. The main contributions of our research are outlined as follows: (i) we developed a stochastic queueing-inventory model for impatient customers on the Jackson network, (ii) to determine the stochastic behavior of the solutions, we deduced the generator matrix, (iii) a stability criterion is constructed, (iv) we derived the stationary distribution of the system using a matrix analytic method, (v) system performance measures are numerically investigated, and finally, (vi) sensitivity analysis is performed to show the effect of changes in the control variables on the total expected cost function of the system. The main objective of this research is to develop a control scheme for generating an effective level of service that will improve the economic performance of the organization and meet future customer demands. Customer impatience (i.e., reneging and jockeying) and perishable inventory control play an important role in increasing capital productivity.

The rest of the article is organized as follows: Section 2: description of the model; Section 3: analysis of the model; Section 4: stability condition; Section 5: steady-state analysis; Section 6: system performance measures; Section 7: cost analysis; Section 8: numerical illustrations; Section 9: sensitivity analysis; and the paper is concluded in Section 10.

2. Description of the Model

In this paper, we consider an open Jackson network of two independent $M/M/1$ parallel queues with continuous review (s_i, S_i) stochastic perishable inventory systems, in which demands arise for individual units by the Poisson manner, where the maximum capacity S_i units for the ith warehouse $(i = 1, 2)$. It is necessary to place an order for $Q_i = ((S_i - s_i);\ i = 1, 2)$ units to bring the inventory level upto S_i if the inventory level falls below or equal to reorder level s_i. We assume that queue-1 is an infinite buffer single-server $M/M/1/\infty$ system with Poisson arrival rate λ_1 and service rate μ_1. The other queue, queue-2 operates as a finite-buffer single-server $M/M/1/C_2$ system with Poisson arrival rate λ_2 and service rate μ_2, where C_2 is the quantity of customers in queue-2.

In this Figure 1, any number of customers are allowed to join queue-1 because of its infinite capacity, and there is limited waiting space in queue-2. When the queue reaches a certain length, further customers are not allowed to join the queue until space becomes available after service completion. If waiting times are intolerable for the arriving customers in the queue-1 or queue-2 then customers may renege (leave) at queue-1 or queue-2 of the system without service with rates ψ_1 and ψ_2, respectively. Also customers may jockey (move) from queue-1 to queue-2 or from queue-2 to queue-1 for leads to economic gains with rates ϕ_1 and ϕ_2,respectively. Service times and the arrival process are mutually independent. The utilization factor of station(queue) i is $\rho_i = (\lambda_i/\mu_i) < 1$. The service discipline at all queues is first-come, first-served (FCFS). A customer in queue i may move to queue j with probability p_{ij} or may leave the system with probability p_{i0} after receiving services. which is defined by

$$p_{i0} = 1 - \sum_{j=1}^{N=2} p_{ij}. \tag{1}$$

2.1. Assumptions of the Model. The following assumptions are used throughout this paper:

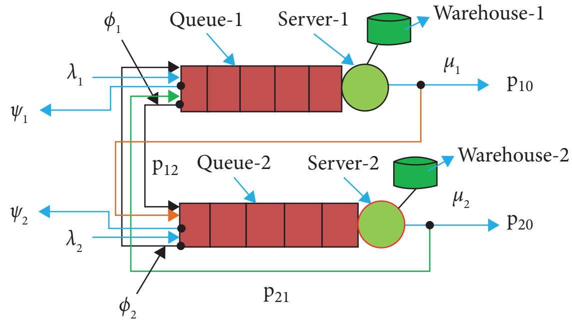

FIGURE 1: Schematic diagram of the proposed model.

(i) The maximum stock level of warehouse i is S_i; $(i = 1, 2)$

(ii) Customer requirements in the system will be met when the inventory level is $I_i(t) \leq S_i$. But customers cannot directly meet their demands when the inventory level is $I_i(t) = 0$, in this case, customers have to wait at least until the next replenishment.

(iii) The replenishment rate is distributed exponentially with the parameter $\theta_i > 0$; $(i = 1, 2)$.

(iv) The perishable rate is exponentially distributed with parameter β_i from the warehouse i; $(i = 1, 2)$.

(v) The service rate in queue i is distributed exponentially with parameter $\mu_i > 0$; $(i = 1, 2)$ and $\mu_1 \neq \mu_2$.

(vi) The arrival rate of a customer at queue i is λ_i; $(i = 1, 2)$ according to the Poisson process.

(vii) The reneging rate is ψ_i from the queue i; $(i = 1, 2)$ and $\psi_1 \neq \psi_2$ in this system, which is exponentially distributed.

(viii) The jockeying rate is exponentially distributed with parameter ϕ_i from queue i, and $\phi_1 \neq \phi_2$ in the system.

(ix) The reneging rates and jockeying rates of the customers in queue i are mutually independent.

2.2. Symbols and Notations. The following notations are used throughout this paper:

$[Q]_{ij}$ denotes the (i, j)th element of a matrix Q

I_k An identity matrix of order k

0 denotes the zero vector

e denotes $(1,1, \ldots, 1)^T$ a column vector of $1'$ s of appropriate order

N denotes the number of queues in the system

$\eta_1 = -(\lambda_1 + \lambda_2 + \theta_1 + \theta_2)$

$\eta_2 = -(\beta_1 + \beta_2 + \mu_2 \mathrm{P}_{20} + \mu_2 \mathrm{P}_{21})$

$\eta_3 = -(\phi_1 + \psi_1 + \mu_1 \mathrm{P}_{12} + \mu_1 \mathrm{P}_{10})$

$\eta_4 = -(\phi_1 + \phi_2 + \psi_2 + \mu_1 \mathrm{P}_{10})$

3. Analysis of the Model

Let $C_1(t), C_2(t), I_1(t), I_2(t)$ symbolize the number of customers in queue-1, the number of customers in queue-2, the inventory level in warehouse-1, and the inventory level in warehouse-2 at time t, respectively. The stochastic process $\{(C_1(t), C_2(t), I_1(t), I_2(t)) \colon t \geq 0\}$ is four-dimensional with state space,

$$\begin{aligned} E &= E_1 \times E_2 \times E_3 \times E_4 \\ &= \{(i_1, j_2, k_1, l_2) \colon i_1 \in E_1, j_2 \in E_2, k_1 \in E_3, l_2 \in E_4\}, \end{aligned} \tag{2}$$

where $E_1 = \{0, 1, 2, \cdots \infty\}$; $E_2 = \{0, 1, \cdots C_2\}$; $E_3 = \{0, 1, \cdots S_1\}$; $E_4 = \{0, 1, \cdots S_2\}$

The infinitesimal generator of this system is described by

$$\tilde{Q} = ((q((i_1, j_2, k_1, l_2), (m_1, n_2, t_1, r_2))), (i_1, j_2, k_1, l_2), (m_1, n_2, t_1, r_2) \in E). \tag{3}$$

It can be achieved by employing the following arguments:

(i) A transition occurs when a customer from outside the system joins queue i; $(i = 1, 2)$.

$$\begin{aligned} (i_1, j_2, k_1, l_2) &\longrightarrow (m_1 = i_1 + 1, n_2 = j_2, t_1 = k_1, r_2 = l_2); \quad 0 \leq i_1 \leq \infty, \\ (i_1, j_2, k_1, l_2) &\longrightarrow (m_1 = i_1, n_2 = j_2 + 1, t_1 = k_1, r_2 = l_2); \quad 0 \leq j_2 \leq C_2. \end{aligned} \tag{4}$$

(ii) When a customer finishes their service in queue i; $(i = 1, 2)$, they can exit the system, reducing the warehouse's inventory by one unit. The system then proceeds from one to the other.

$$\begin{aligned}
&(i_1, j_2, k_1, l_2) \longrightarrow (m_1 = i_1 - 1, n_2 = j_2, t_1 = k_1 - 1, r_2 = l_2); 1 \le i_1 \le \infty;\ 1 \le k_1 \le S_1,\\
&(i_1, j_2, k_1, l_2) \longrightarrow (m_1 = i_1, n_2 = j_2 - 1, t_1 = k_1, r_2 = l_2 - 1); 1 \le j_2 \le C_2;\ 1 \le l_2 \le S_2.
\end{aligned} \tag{5}$$

(iii) A customer who completes service in queue i can move to queue j; $(i \ne j;\ i, j = 1, 2)$ and the inventory level of the kth warehouse $(k = 1, 2)$ will be reduced by one item. As a result, a transition from queue i to queue j is possible.

$$\begin{aligned}
&(i_1, j_2, k_1, l_2) \longrightarrow (m_1 = i_1 - 1, n_2 = j_2 + 1, t_1 = k_1 - 1, r_1 = l_2);\ 1 \le i_1 \le \infty;\ 1 \le k_1 \le S_1,\\
&(i_1, j_2, k_1, l_2) \longrightarrow (m_1 = i_1 + 1, n_2 = j_2 - 1, t_1 = k_1, r_2 = l_2 - 1);\ 1 \le j_2 \le C_2;\ 1 \le l_2 \le S_2.
\end{aligned} \tag{6}$$

(iv) Inventory levels will be modified in the warehouse k; $(k = 1, 2)$ due to the perishability of the items. The system then switches from one to the other.

$$\begin{aligned}
&(i_1, j_2, k_1, l_2) \longrightarrow (m_1 = i_1, n_2 = j_2, t_1 = k_1 - 1, r_2 = l_2);\ 1 \le k_1 \le S_1,\\
&(i_1, j_2, k_1, l_2) \longrightarrow (m_1 = i_1, n_2 = j_2, t_1 = k_2, r_2 = l_2 - 1);\ 1 \le l_2 \le S_2.
\end{aligned} \tag{7}$$

(v) The inventory levels in the warehouse k; $(k = 1, 2)$ will be increased due to the replenishment of the Q_k products. The system then switches from one mode to another.

$$\begin{aligned}
&(i_1, j_2, k_1, l_2) \longrightarrow (m_1 = i_1, n_2 = j_2, t_1 = k_1 + Q_1, r_2 = l_2);\ 0 \le k_1 \le s_1,\\
&(i_1, j_2, k_1, l_2) \longrightarrow (m_1 = i_1, n_2 = j_2, t_1 = k_1, r_2 = l_2 + Q_2);\ 0 \le l_2 \le s_2.
\end{aligned} \tag{8}$$

(vi) For reneging, the impatient customer exits the queue i; $(i = 1, 2)$ without receiving service and performs a transition, which reduces the size of the customer in the waiting space by one unit.

$$\begin{aligned}
&(i_1, j_2, k_1, l_2) \longrightarrow (m_1 = i_1 - 1, n_2 = j_2, t_1 = k_1, r_2 = l_2);\ 1 \le i_1 \le \infty,\\
&(i_1, j_2, k_1, l_2) \longrightarrow (m_1 = i_1, n_2 = j_2 - 1, t_1 = k_1, r_2 = l_2);\ 1 \le j_2 \le C_2.
\end{aligned} \tag{9}$$

(vii) For jockeying, the impatience customer at queue i can move to queue j; $(i \ne j; i, j = 1, 2)$ for getting quick service from the shortest queue of the system. Hence, the system makes a transition from

$$\begin{aligned}
&(i_1, j_2, k_1, l_2) \longrightarrow (m_1 = i_1 - 1, n_2 = j_2 + 1, t_1 = k_1, r_2 = l_2);\ 1 \le i_1 \le \infty,\\
&(i_1, j_2, k_1, l_2) \longrightarrow (m_1 = i_1 + 1, n_2 = j_2 - 1, t_1 = k_1, r_2 = l_2);\ 1 \le j_2 \le C_2.
\end{aligned} \tag{10}$$

(viii) For other transitions from (i_1, j_2, k_1, l_2) to state (m_1, n_2, t_1, r_2), except when $(i_1, j_2, k_1, l_2) \ne (m_1, n_2, t_1, r_2)$, the rate is zero.

(ix) Finally, we notice that the diagonal entry is equal to the negative of the sum of the other entries in that row. More explicitly

$$\gamma = q\left(\left(i_1, j_2, k_1, l_2\right), \left(m_1, n_2, t_1, r_2\right)\right) = -\sum_{i_1}\sum_{j_2}\sum_{k_1}\sum_{l_2} a\left(\left(i_1, j_2, k_1, l_2\right), \left(m_1, n_2, t_1, r_2\right)\right)_{\left(i_1, j_2, k_1, l_2\right) \neq \left(m_1, n_2, t_1, r_2\right)}. \tag{11}$$

Hence, we have

$$\widetilde{Q} = q\left(\left(i_1, j_2, k_1, l_2\right), \left(m_1, n_2, t_1, r_2\right)\right) = \begin{cases} \lambda_1, & \begin{array}{ll} m_1 = i_1 + 1, i_1 = 0, 1, \ldots; & n_2 = j_2, j_2 = 0, 1, \ldots, C_2 \\ t_1 = k_1, k_1 = 0, 1, \ldots, S_1; & r_2 = l_2, l_2 = 0, 1, \ldots, S_2 \end{array} \\ \lambda_2, & \begin{array}{ll} m_1 = i_1, i_1 = 0, 1, \ldots; & n_2 = j_2 + 1, j_2 = 0, 1, \ldots, C_2 \\ t_1 = k_1, k_1 = 0, 1, \ldots, S_1; & r_2 = l_2, l_2 = 0, 1, \ldots, S_2 \end{array} \\ \mu_1 p_{10}, & \begin{array}{ll} m_1 = i_1 - 1, i_1 = 1, \ldots; & n_2 = j_2, j_2 = 0, 1, \ldots, C_2 \\ t_1 = k_1 - 1, k_1 = 1, \ldots, S_1; & r_2 = l_2, l_2 = 0, 1, \ldots, S_2 \end{array} \\ \mu_2 p_{20}, & \begin{array}{ll} m_1 = i_1, i_1 = 0, 1, \ldots; & n_2 = j_2 - 1, j_2 = 1, \ldots, C_2 \\ t_1 = k_1, k_1 = 0, 1, \ldots, S_1; & r_2 = l_2 - 1, l_2 = 1, \ldots, S_2 \end{array} \\ \mu_1 p_{12}, & \begin{array}{ll} m_1 = i_1 - 1, i_1 = 1, \ldots; & n_2 = j_2 + 1, j_2 = 0, 1, \ldots, C_2 \\ t_1 = k_1 - 1, k_1 = 1, \ldots, S_1; & r_2 = l_2, l_2 = 0, 1, \ldots, S_2 \end{array} \\ \mu_2 p_{21}, & \begin{array}{ll} m_1 = i_1 + 1, i_1 = 0, 1, \ldots; & n_2 = j_2 - 1, j_2 = 1, \ldots, C_2 \\ t_1 = k_1, k_1 = 0, 1, \ldots, S_1; & r_2 = l_2 - 1, l_2 = 1, \ldots, S_2 \end{array} \\ \beta_1, & \begin{array}{ll} m_1 = i_1, i_1 = 0, 1, \ldots; & n_2 = j_2, j_2 = 0, 1, \ldots, C_2 \\ t_1 = k_1 - 1, k_1 = 1, \ldots, S_1; & r_2 = l_2, l_2 = 0, 1, \ldots, S_2 \end{array} \\ \beta_2, & \begin{array}{ll} m_1 = i_1, i_1 = 0, 1, \ldots; & n_2 = j_2, j_2 = 0, 1, \ldots, C_2 \\ t_1 = k_1, k_1 = 0, 1, \ldots, S_1; & r_2 = l_2 - 1, l_2 = 1, \ldots, S_2 \end{array} \\ \theta_1, & \begin{array}{ll} m_1 = i_1, i_1 = 0, 1, \ldots; & n_2 = j_2, j_2 = 0, 1, \ldots, C_2 \\ t_1 = k_1 + Q_1, k_1 = 0, 1, \ldots, s_1; & r_2 = l_2, l_2 = 0, 1, \ldots, S_2 \end{array} \\ \theta_2, & \begin{array}{ll} m_1 = i_1, i_1 = 0, 1, \ldots; & n_2 = j_2, j_2 = 0, 1, \ldots, C_2 \\ t_1 = k_1, k_1 = 0, 1, \ldots, S_1; & r_2 = l_2 + Q_2, l_2 = 0, 1, \ldots, s_2 \end{array} \\ \psi_1, & \begin{array}{ll} m_1 = i_1 - 1, i_1 = 0, 1, \ldots; & n_2 = j_2, j_2 = 0, 1, \ldots, C_2 \\ t_1 = k_1, k_1 = 0, 1, \ldots, S_1; & r_2 = l_2, l_2 = 0, 1, \ldots, S_2 \end{array} \\ \psi_2, & \begin{array}{ll} m_1 = i_1, i_1 = 0, 1, \ldots; & n_2 = j_2 - 1, j_2 = 1_2, \ldots, C_2 \\ t_1 = k_1, k_1 = 0, 1, \ldots, S_1; & r_2 = l_2, l_2 = 0, 1, \ldots, S_2 \end{array} \\ \phi_1, & \begin{array}{ll} m_1 = i_1 - 1, i_1 = 1, \ldots; & n_2 = j_2 + 1, j_2 = 0, 1, \ldots, C_2 \\ t_1 = k_1, k_1 = 0, 1, \ldots, S_1; & r_2 = l_2, l_2 = 0, 1, \ldots, S_2 \end{array} \\ \phi_2, & \begin{array}{ll} m_1 = i_1 + 1, i_1 = 0, 1, \ldots; & n_2 = j_2 - 1, j_2 = 1, \ldots, C_2 \\ t_1 = k_1, k_1 = 0, 1, \ldots, S_1; & r_2 = l_2, l_2 = 0, 1, \ldots, S_2 \end{array} \\ \gamma, & \begin{array}{ll} m_1 = i_1, i_1 = 0, 1, \ldots; & n_2 = j_2, j_2 = 0, 1, \ldots, C_2 \\ t_1 = k_1, k_1 = 0, 1, \ldots, S_1; & r_2 = l_2, l_2 = 0, 1, \ldots, S_2 \end{array} \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

Define $\widetilde{Q} = (Q_{i_1 m_1})$, where $Q_{i_1 m_1} = ((q((i_1, j_2, k_1, l_2), (m_1, n_2, t_1, r_2))), (i_1, j_2, k_1, l_2), (m_1, n_2, t_1, r_2) \in E$

The infinitesimal generator $\widetilde{Q}$ can be conveniently written in a block-partitioned matrix with entries

$$\widetilde{Q} = \left(Q_{i_1 m_1}\right) = \begin{cases} D_0; & m_1 = i_1 \quad i_1 = 0 \\ G_0; & m_1 = i_1 + 1 \quad \forall i_1 = 0, 1, 2, \cdots \\ G_1; & m_1 = i_1 \quad \forall i_1 = 1, 2, \cdots \\ G_2; & m_1 = i_1 - 1 \quad \forall i_1 = 1, 2, \cdots \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

More explicitly,

$$\widetilde{Q} = \begin{bmatrix} D_0 & G_0 & 0 & 0 & 0 & \cdots \\ G_2 & G_1 & G_0 & 0 & 0 & \cdots \\ 0 & G_2 & G_1 & G_0 & 0 & \cdots \\ 0 & 0 & G_2 & G_1 & G_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{14}$$

where the blocks D_0 and $G_{i_1}; i_1 = 0, 1, 2$ are squares matrices, each order is $((Q_1 + C_2)(Q_1 + S_2Q_2)) \times ((Q_1 + C_2)(Q_1 + S_2Q_2))$, D_0 represents the transition within the level $i_1, i_1 \geq 0, G_0$ represents the transition from level i_1 to $i_1 + 1, i_1 \geq 0, G_2$ represents the transition from level i_1 to $i_1 - 1, i_1 \geq 1, G_1$ represents the transition within the level $i_1 \geq 1$.

D_0 is a $(Q_1 + Q_2) \times (Q_1 + Q_2)$ submatrix and which is given by

$$[D_0]_{i_1 j_2} = \begin{cases} D_{i_1}; & j_2 = i_1 \quad i_1 = 1, 2, \ldots, S_1 + 1 \\ D_5; & j_2 = i_1 + 1 \quad i_1 = s_1, Q_1 + 1 \\ D_6; & j_2 = i_1 + 2 \quad i_1 = s_1, Q_1 \\ D_7; & j_2 = i_1 - 1 \quad i_1 = Q_1, S_1 + 1 \\ D_8; & j_2 = i_1 - 2 \quad i_1 = S_1, S_1 + 1 \\ 0, & \text{otherwise,} \end{cases} \tag{15}$$

where all square submatrices of D_0 are the same order, each order is $(Q_1 (Q_2 + C_2) \times Q_1 (Q_2 + C_2))$.

$$D_1 = \begin{cases} \beta_1, & m_1 = i_1, i_1 = 0 \quad n_2 = j_2, j_2 = 0 \\ & t_1 = k_1 - 1, k_1 = s_1; \quad r_2 = l_2, l_2 = 0, 1, \ldots, S_2. \\ \beta_2, & m_1 = i_1, i_1 = 0 \quad n_2 = j_2, j_2 = 0 \\ & t_1 = k_1, k_1 = s_1 - 1, s_1; \quad r_2 = l_2 - 1, l_2 = 1, \ldots, S_2. \\ \theta_2, & m_1 = i_1, i_1 = 0 \quad n_2 = j_2, j_2 = 0 \\ & t_1 = k_1, k_1 = s_1 - 1, s_1; \quad r_2 = l_2 + Q_2, l_2 = s_2 - 1, s_2; \\ \eta_1, & m_1 = i_1, i_1 = 0 \quad n_2 = j_2, j_2 = 0 \\ & t_1 = k_1, k_1 = s_1 - 1; \quad r_2 = l_2, l_2 = s_2 - 1 \\ \eta_1 - \beta_2, & m_1 = i_1, i_1 = 0 \quad n_2 = j_2, j_2 = 0 \\ & t_1 = k_1, k_1 = s_1 - 1; \quad r_2 = l_2, l_2 = s_2 \\ \eta_1 + \theta_2 - \beta_2, & m_1 = i_1, i_1 = 0 \quad n_2 = j_2, j_2 = 0 \\ & t_1 = k_1, k - 1 = s_1 - 1; \quad r_2 = l_2, l_2 = s_2 + 1, S_2 \\ \eta_1 - \beta_1, & m_1 = i_1, i_1 = 0 \quad n_2 = j_2, j_2 = 0 \\ & t_1 = k_1, k_1 = s_1; \quad r_2 = l_2, l_2 = s_2 - 1 \\ \eta_1 - \beta_1 - \beta_2, & m_1 = i_1, i_1 = 0 \quad n_2 = j_2, j_2 = 0 \\ & t_1 = k_1, k_1 = s_1; \quad r_2 = l_2, l_2 = s_2 \\ \eta_1 - \beta_1 - \beta_2 + \theta_2, & m_1 = i_1, i_1 = 0 \quad n_2 = j_2, j_2 = 0 \\ & t_1 = k_1, k_1 = s_1; \quad r_2 = l_2, l_2 = s_2 + 1, S_2 \\ 0, & \text{otherwise} \end{cases}$$

$$D_2 = \begin{cases} \eta_1 + \theta_1 - \beta_1, & \begin{matrix} m_1 = i_1, i_1 = 0 & n_2 = j_2, j_2 = 0 \\ t_1 = k_1, k_1 = s_1 + 1, S_1; & r_2 = l_2, l_2 = s_2 - 1 \end{matrix} \\ \eta_1 + \theta_1 - \beta_1 - \beta_2, & \begin{matrix} m_1 = i_1, i_1 = 0 & n_2 = j_2, j_2 = 0 \\ t_1 = k_1, k_1 = s_1 + 1, S_1; & r_2 = l_2, l_2 = s_2 \end{matrix} \\ -(\beta_1 + \beta_2 + \lambda_1 + \lambda_2), & \begin{matrix} m_1 = i_1, i_1 = 0 & n_2 = j_2, j_2 = 0 \\ t_1 = k_1, k_1 = s_1 + 1, S_1; & r_2 = l_2, l_2 = s_2 + 1, S_2 \end{matrix} \\ \beta_1, & \begin{matrix} m_1 = i_1, i_1 = 0 & n_2 = j_2, j_2 = 0 \\ t_1 = k_1 - 1, k_1 = S_1; & r_2 = l_2, l_2 = 0, 1, \ldots, S_2 \end{matrix} \\ \beta_2, & \begin{matrix} m_1 = i_1, i_1 = 0 & n_2 = j_2, j_2 = 0 \\ t_1 = k_1, k_1 = s_1 + 1, S_1; & r_2 = l_2 - 1, l_2 = 1, \ldots, S_2 \end{matrix} \\ \theta_2, & \begin{matrix} m_1 = i_1, i_1 = 0 & n_2 = j_2, j_2 = 0 \\ t_1 = k_1, k_1 = s_1 + 1, S_1; & r_2 = l_2 + Q_2, l_2 = s_2 - 1, s_2 \end{matrix} \\ 0, & \text{otherwise} \end{cases}$$

$$\mathrm{D}_3 = \begin{cases} \beta_1, & \begin{matrix} m_1 = i_1, i_1 = 0 & n_2 = j_2, j_2 = 1 \\ t_1 = k_1 - 1, k_1 = s_1; & r_2 = l_2, l_2 = 0, 1, \ldots, S_2 \end{matrix} \\ \beta_2, & \begin{matrix} m_1 = i_1, i_1 = 0 & n_2 = j_2, j_2 = 1 \\ t_1 = k_1, k_1 = s_1 - 1, s_1; & r_2 = l_2 - 1, l_2 = 1, \ldots, S_2 \end{matrix} \\ \theta_2, & \begin{matrix} m_1 = i_1, i_1 = 0 & n_2 = j_2, j_2 = 1 \\ t_1 = k_1, k_1 = s_1 - 1, s_1; & r_2 = l_2 + Q_2, l_2 = s_2 - 1, s_2 \end{matrix} \\ \eta_1 + \lambda_2, & \begin{matrix} m_1 = i_1, i_1 = 0 & n_2 = j_2, j_2 = 1 \\ t_1 = k_1, k_1 = s_1 - 1; & r_2 = l_2, l_2 = s_2 - 1 \end{matrix} \\ \eta_1 + \lambda_2 + \beta_1 + \eta_2, & \begin{matrix} m_1 = i_1, i_1 = 0 & n_2 = j_2, j_2 = 1 \\ t_1 = k_1, k_1 = s_1 - 1; & r_2 = l_2, l = s_2 \end{matrix} \\ \beta_1 + \eta_1 - \theta_1 - \lambda_1, & \begin{matrix} m_1 = i_1, i_1 = 0 & n_2 = j_2, j_2 = 1 \\ t_1 = k_1, k_1 = s_1 - 1; & r_2 = l_2, l_2 = s_2 + 1, S_2 \end{matrix} \\ \eta_1 - \beta_1 + \lambda_2, & \begin{matrix} m_1 = i_1, i_1 = 0 & n_2 = j_2, j_2 = 1 \\ t_1 = k_1, k_1 = s_1; & r_2 = l_2, l_2 = s_2 - 1 \end{matrix} \\ \eta_1 + \eta_2 + \lambda_2 - \phi_2 - \psi_2, & \begin{matrix} m_2 = i_2, i_2 = 0 & n_2 = j_2, j_2 = 1 \\ t_1 = k_1, k_1 = s_1; & r_2 = l_2, l_2 = s_2 \end{matrix} \\ \eta_2 - \lambda_1 - \theta_1 - \phi_2 - \psi_2, & \begin{matrix} m_1 = i_1, i_1 = 0 & n_2 = j_2, j_2 = 1 \\ t_1 = k_1, k_1 = s_1; & r_2 = l_2, l_2 = s_2 + 1, S_2 \end{matrix} \\ 0, & \text{otherwise} \end{cases}$$

$$D_4 = \begin{cases} \beta_1, & \begin{array}{ll} m_1 = i_1, i_1 = 0 & n_2 = j_2, j_2 = 1 \\ t_1 = k_1 - 1, k_1 = S_1; & r_2 = l_2, l_2 = 0, 1, \ldots, S_2 \end{array} \\ \beta_2, & \begin{array}{ll} m_1 = i_1, i_1 = 0 & n_2 = j_2, j_2 = 1 \\ t_1 = k_1, k_1 = S_1 - 1, S_1; & r_2 = l_2 - 1, l_2 = 1, \ldots, S_2 \end{array} \\ \theta_2, & \begin{array}{ll} m_1 = i_1, i_1 = 0 & n_2 = j_2, j_2 = 1 \\ t_1 = k_1, k_1 = S_1 - 1, S_1; & r_2 = l_2 + Q_2, l_2 = s_2 - 1, s_2 \end{array} \\ -(\lambda_1 + \beta_1 + \theta_2), & \begin{array}{ll} m_1 = i_1, i_1 = 0 & n_2 = j_2, j_2 = 1 \\ t_1 = k_1, k_1 = s_1 + 1, S_2, ; & r_2 = l_2, l_2 = s_2 - 1 \end{array} \\ \eta_2 - \lambda_1 - \theta_2 - \phi_2 - \psi_2, & \begin{array}{ll} m_1 = i_1, i_1 = 0 & n_2 = j_2, j_2 = 1 \\ t_1 = k_1, k_1 = s_1 + 1, S_2, ; & r_2 = l_2, l_2 = s_2 \end{array} \\ -\lambda_1 + \eta_2 - \phi_2 - \psi_2, & \begin{array}{ll} m_1 = i_1, i_1 = 0 & n_2 = j_2, j_2 = 1 \\ t_1 = k_1, k_1 = s_1 + 1, S_1, ; & r_2 = l_2, l_2 = s_2 + 1, S_2 \end{array} \\ 0, & \text{otherwise} \end{cases}$$

$$D_5 = \mathrm{di\,ag}\left(\theta_1, \theta_1, \ldots, \theta_1\right);$$

$$D_6 = \mathrm{diag}\left(\lambda_2, \lambda_2, \ldots, \lambda_2\right)$$

$$D_7 = \begin{cases} \beta_1, & m_1 = i_1, i_1 = 0 \quad n_2 = j_2, j_2 = 0 \\ & t_1 = k_1 - 1, k_1 = S_1 - 1; \quad r_2 = l_2, l_2 = 0, 1, \ldots, S_2. \\ 0, & \text{otherwise} \end{cases}$$

$$D_8 = \begin{cases} \mu_2 p_{20}, & m_1 = i_1, i_1 = 0 \quad n_2 = j_2 - 1, j_2 = 1 \\ & t_1 = k_1, k_1 = 0, 1; \quad r_2 = l_2 - 1, l_2 = 1, \ldots, S_2. \\ 0, & \text{otherwise.} \end{cases} \tag{16}$$

For level $i_1 = 1, 2, \ldots;$

$$G_2 = \begin{cases} \psi_1, & \begin{array}{ll} m_1 = i_1 - 1, & n_2 = j_2, j_2 = 0, 1, \ldots, C_2 \\ t_1 = k_1, k_1 = 1, \ldots, S_1; & r_2 = l_2, l_2 = 0, 1, \ldots, S_2. \end{array} \\ \mu_1 p_{10}, & \begin{array}{ll} m_1 = i_1 - 1, & n_2 = j_2, j_2 = 0, 1, \ldots, C_2 \\ t_1 = k_1 - 1, k_1 = 1, \ldots, S_1; & r_2 = l_2, l_2 = 0, 1, \ldots, S_2. \end{array} \\ \mu_1 p_{12}, & \begin{array}{ll} m_1 = i_1 - 1, & n_2 = j_2 + 1, j_2 = 0, 1, \ldots, C_2 \\ t_1 = k_1 - 1, k_1 = 1, \ldots, S_1; & r_2 = l_2, l_2 = 0, 1, \ldots, S_2. \end{array} \\ \phi_1, & \begin{array}{ll} m_1 = i_1 - 1, & n_2 = j_2 + 1, j_2 = 0, 1, \ldots, C_2 \\ t_1 = k_1, k_1 = 1, \ldots, S_1; & r_2 = l_2, l_2 = 0, 1, \ldots, S_2. \end{array} \\ 0, & \text{otherwise.} \end{cases} \tag{17}$$

For level $i_1 = 0, 1, 2, \ldots;$

$$G_0 = \begin{cases} \lambda_1, & \begin{matrix} m_1 = i_1 + 1, & n_2 = j_2, j_2 = 0, 1, \ldots, C_2 \\ t_1 = k_1, k_1 = 0, 1, \ldots, S_1; & r_2 = l_2, l - 2 = 0, 1, \ldots, S_2. \end{matrix} \\ \mu_2 p_{21}, & \begin{matrix} m_1 = i_1 + 1, & n_2 = j_2 - 1, j_2 = 1, \ldots, C_2 \\ t_1 = k_1, k_1 = 0, 1, \ldots, S_1; & r_2 = l_2 - 1, l_2 = 1, \ldots, S_2. \end{matrix} \\ \phi_2, & \begin{matrix} m_1 = i_1 + 1, & n_2 = j_2 - 1, j_2 = 1, \ldots, C_2 \\ t_1 = k_1, k_1 = 0, 1, \ldots, S_1; & r_2 = l_2, l_2 = 1, \ldots, S_2. \end{matrix} \\ 0, & \text{otherwise.} \end{cases} \tag{18}$$

where G_1 is a $(Q_1 + C_2) \times (Q_1 + C_2)$ submatrix, which are given for level $i_1 = 1, 2, \ldots;$

$$[G_1]^{i_1}_{k_1 l_2} = \begin{cases} G_k; & l_2 = k_1 & k_1 = 3, 4, \ldots, S_1 + 3 \\ G_7; & l_2 = k_1 + 1 & k_1 = s_1, Q_1 + 1 \\ G_8; & l_2 = k_1 + 2 & k_1 = s_1, Q_1 \\ G_9; & l_2 = k_1 - 1 & k_1 = Q_1, S_1 + 1 \\ G_{10}; & l_2 = k_1 - 2 & k_1 = S_1, S_1 + 1 \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

It may be noted that, all the submatrices of G_1 are same order, which is of order $(Q_1 Q_2)(S_1 Q_1 + Q_2) \times (Q_1 Q_2)(S_1 Q_1 + Q_2)$.

For level $i_1 = 1, 2, \ldots$

$$G_3 = \begin{cases} \beta_1, & \begin{matrix} m_1 = i_1, & n_2 = j_2, j_2 = 0 \\ t_1 = k_1 - 1, k_1 = s_1; & r_2 = l_2, l_2 = 0, 1, \ldots, S_2. \end{matrix} \\ \beta_2, & \begin{matrix} m_1 = i_1, & n_2 = j_2, j_2 = 0 \\ t_1 = k_1, k_1 = s_1 - 1, s_1; & r_2 = l_2 - 1, l_2 = 1, \ldots, S_2. \end{matrix} \\ \theta_2, & \begin{matrix} m_1 = i_1, & n_2 = j_2, j_2 = 0 \\ t_1 = k_1, k_1 = s_1 - 1, s_1; & r_2 = l_2 + Q_2, l_2 = s_2 - 1, s_2; \end{matrix} \\ \eta_1, & \begin{matrix} m_1 = i_1, & n_2 = j_2, j_2 = 0 \\ t_1 = k_1, k_1 = s_1 - 1; & r_2 = l_2, l_2 = s_2 - 1 \end{matrix} \\ \eta_1 - \beta_2, & \begin{matrix} m_1 = i_1, & n_2 = j_2, j_2 = 0 \\ t_1 = k_1, k_1 = s_1 - 1; & r_2 = l_2, l_2 = s_2 \end{matrix} \\ \eta_1 + \theta_2 - \beta_2, & \begin{matrix} m_1 = i_1, & n_2 = j_2, j_2 = 0 \\ t_1 = k_1, k_1 = s_1 - 1; & r_2 = l_2, l_2 = s_2 + 1, S_2 \end{matrix} \\ \eta_1 - \beta_1 + \eta_3 + \phi_1, & \begin{matrix} m_1 = i_1, & n_2 = j_2, j_2 = 0 \\ t_1 = k_1, k_1 = s_1; & r_2 = l_2, l_2 = s_2 - 1 \end{matrix} \\ \eta_1 - \beta_1 - \beta_2 + \eta_3, & \begin{matrix} m_1 = i_1, & n_2 = j_2, j_2 = 0 \\ t_1 = k_1, k_1 = s_1; & r_2 = l_2, l_2 = s_2 \end{matrix} \\ \eta_1 - \beta_1 - \beta_2 + \theta_2 + \eta_3, & \begin{matrix} m_1 = i_1, & n_2 = j_2, j_2 = 0 \\ t_1 = k_1, k_1 = s_1; & r_2 = l_2, l_2 = s_2 + 1, S_2 \end{matrix} \\ 0, & \text{otherwise.} \end{cases} \tag{20}$$

For level $i_1 = 1, 2, \ldots$

$$G_4 = \begin{cases} \eta_1 + \theta_1 - \beta_1 + \eta_3, & m_1 = i_1, \quad n_2 = j_2, j_2 = 0 \\ & t_1 = k_1, k_1 = s_1 + 1, S_1; \quad r_2 = l_2, l_2 = s_2 - 1 \\ \eta_1 + \theta_1 - \beta_1 - \beta_2 + \eta_3, & m_1 = i_1, \quad n_2 = j_2, j_2 = 0 \\ & t_1 = k_1, k_1 = s_1 + 1, S_1; \quad r_2 = l_2, l_2 = s_2 \\ -(\beta_1 + \beta_2 + \lambda_1 + \lambda_2) + \eta_3, & m_1 = i_1, \quad n_2 = j_2, j_2 = 0 \\ & t_1 = k_1, k_1 = s_1 + 1, S_1; \quad r_2 = l_2, l_2 = s_2 + 1, S_2 \\ \beta_1, & m_1 = i_1, \quad n_2 = j_2, j_2 = 0 \\ & t_1 = k_1 - 1, k_1 = S_1; \quad r_2 = l_2, l_2 = 0, 1, \ldots, S_2 \\ \beta_2, & m_1 = i_1, \quad n_2 = j_2, j_2 = 0 \\ & t_1 = k_1, k_1 = s_1 + 1, S_1; \quad r_2 = l_2 - 1, l_2 = 1, \ldots, S_2 \\ \theta_2, & m_1 = i_1, \quad n_2 = j_2, j_2 = 0 \\ & t_1 = k_1, k_1 = s_1 + 1, S_1; \quad r_2 = l_2 + Q_2, l_2 = s_2 - 1, s_2 \\ 0, & \text{otherwise.} \end{cases} \tag{21}$$

For level $i_1 = 1, 2, \ldots$

$$G_5 = \begin{cases} \beta_1, & m_1 = i_1, \quad n_2 = j_2, j_2 = 1 \\ & t_1 = k_1 - 1, k_1 = s_1; \quad r_2 = l_2, l_2 = 0, 1, \ldots, S_2 \\ \beta_2, & m_1 = i_1, \quad n_2 = j_2, j_2 = 1 \\ & t_1 = k_1, k_1 = s_1 - 1, s_1; \quad r_2 = l_2 - 1, l_2 = 1, \ldots, S_2 \\ \theta_2, & m_1 = i_1, \quad n_2 = j_2, j_2 = 1 \\ & t_1 = k_1, k_1 = s_1 - 1, s_1; \quad r_2 = l_2 + Q_2, l_2 = s_2 - 1, s_2 \\ \eta_1 + \lambda_2, & m_1 = i_1, \quad n_2 = j_2, j_2 = 1 \\ & t_1 = k_1, k_1 = s_1 - 1; \quad r_2 = l_2, l_2 = s_2 - 1 \\ \eta_1 + \lambda_2 + \beta_1 + \eta_2 - \phi_2 - \psi_2, & m_1 = i_1, \quad n_2 = j_2, j_2 = 1 \\ & t_1 = k_1, k_1 = s_1 - 1; \quad r_2 = l_2, l_2 = s_2 \\ \beta_1 + \eta_1 - \theta_1 - \lambda_1 + \eta_2 - \phi_2 - \psi_2, & m_1 = i_1, \quad n_2 = j_2, j_2 = 1 \\ & t_1 = k_1, k_1 = s_1 - 1; \quad r_2 = l_2, l_2 = s_2 + 1, S_2 \\ \eta_1 - \beta_1 + \lambda_2 - \mu_1 p_{10} - \psi_1, & m_1 = i_1, \quad n_2 = j_2, j_2 = 1 \\ & t_1 = k_1, k_1 = s_1; \quad r_2 = l_2, l_2 = s_2 - 1 \\ \eta_1 + \eta_2 + \lambda_2 + \eta_4, & m_1 = i_1, \quad n_2 = j_2, j_2 = 1 \\ & t_1 = k_1, k_1 = s_1; \quad r_2 = l_2, l_2 = s_2 \\ \eta_2 - \lambda_1 - \theta_1 + \eta_4, & m_1 = i_1, \quad n_2 = j_2, j_2 = 1 \\ & t_1 = k_1, k_1 = s_1; \quad r_2 = l_2, l_2 = s_2 + 1, S_2 \\ 0, & \text{otherwise.} \end{cases} \tag{22}$$

For level

$$i_1 = 1, 2, \ldots$$

$$G_6 = \begin{cases} \beta_1, & m_1 = i_1, \quad n_2 = j_2, j_2 = 1 \\ & t_1 = k_1 - 1, k_1 = S_1; \quad r_2 = l_2, l_2 = 0, 1, \ldots, S_2 \\ \beta_2, & m_1 = i_1, \quad n_2 = j_2, j_2 = 1 \\ & t_1 = k_1, k_1 = S_1 - 1, S_1; \quad r_2 = l_2 - 1, l_2 = 1, \ldots, S_2 \\ \theta_2, & m_1 = i_1, \quad n_2 = j_2, j_2 = 1 \\ & t_1 = k_1, k_1 = S_1 - 1, S_1; \quad r_2 = l_2 + Q_2, l_2 = s_2 - 1, s_2 \\ -(\lambda_1 + \beta_1 + \theta_2 + \mu_1 p_{10} + \psi_1), & m_1 = i_1, \quad n_2 = j_2, j_2 = 1 \\ & t_1 = k_1, k_1 = s_1 + 1, S_2, ; \quad r_2 = l_2, l_2 = s_2 - 1 \\ \eta_2 - \lambda_1 - \theta_2 + \eta_4, & m_1 = i_1, \quad n_2 = j_2, j_2 = 1 \\ & t_1 = k_1, k_1 = s_1 + 1, S_2, ; \quad r_2 = l_2, l_2 = s_2 \\ -\lambda_1 + \eta_2 + \eta_2, & m_1 = i_1, \quad n_2 = j_2, j_2 = 1 \\ & t_1 = k_1, k_1 = s_1 + 1, S_1, ; \quad r_2 = l_2, l_2 = s_2 + 1, S_2 \\ 0, & \text{otherwise.} \end{cases} \tag{23}$$

For level $i_1 = 1, 2, \ldots$

$$G_7 = \begin{cases} \theta_1, & m_1 = i_1, \quad n_2 = j_2, j_2 = 0 \\ & t_1 = k_1 + Q_1, k_1 = s_1 - 1, s_1; \quad r_2 = l_2, l_2 = 0, 1, \ldots, S_2 \\ 0, & \text{otherwise.} \end{cases} \tag{24}$$

For level $i_1 = 1, 2, \ldots$

$$G_8 = \begin{cases} \lambda_2, & m_1 = i_1, \quad n_2 = j_2 + 1, j_2 = 0, \\ & t_1 = k_1, k_1 = s_1 - 1, s_1; \quad r_2 = l_2, l_2 = 0, 1, \ldots, S_2, \\ 0, & \text{otherwise.} \end{cases} \tag{25}$$

For level $i_1 = 1, 2, \ldots$

$$G_9 = \begin{cases} \beta_1, & m_1 = i_1, \quad n_2 = j_2, j_2 = 0, \\ & t_1 = k_1 - 1, k_1 = S_1 - 1; \quad r_2 = l_2, l_2 = 0, 1, \ldots, S_2, \\ 0, & \text{otherwise,} \end{cases} \tag{26}$$

For level $i_1 = 1, 2, \ldots$

$$G_{10} = \begin{cases} \mu_2 p_{20}, & m_1 = i_1, \quad n_2 = j_2 - 1, j_2 = 1, \\ & t_1 = k_1, k - 1 = 0, 1; \quad r_2 = l_2 - 1, l_2 = 1, \ldots, S_2, \\ \psi_2, & m_1 = i_1, \quad n_2 = j_2 - 1, j_2 = 1, \\ & t_1 = k_1, k_1 = 0, 1; \quad r_2 = l_2, l_2 = 1, \ldots, S_2, \\ 0, & \text{otherwise.} \end{cases} \tag{27}$$

4. Stability Condition

Let Π be the steady-state probability vector of the generator $G = G_0 + G_1 + G_2$. In other words, Π satisfies $\Pi G = 0$ and $\Pi e = 1$, where e is the column vector of 1 with the appropriate order. The Markov chain is considered a level-independent quasi-birth-death process. The QBD shape of the model shows that the queuing system is stable if and only if [21]

$$\pi G_0 e < \pi G_2 e. \tag{28}$$

We can establish the following stability condition using (28) for this model:

$$\begin{aligned}&\lambda_1 \sum_{i_1=0}^{C_2-2}\sum_{j_2=0}^{Q_2-1}\sum_{k_1=0}^{S_1}\sum_{l_2=0}^{S_2}\Pi^{(i_1,j_2,k_1,l_2)} + (\mu_2 p_{21} + \phi_2)\sum_{i_1=0}^{C_2-2}\sum_{j_2=1}^{Q_2-1}\sum_{k_1=0}^{S_1}\sum_{l_2=1}^{S_2}\Pi^{(i_1,j_2,k_1,l_2)}\\ &< (\mu_1 p_{10} + \psi_1)\sum_{i_1=1}^{C_2-1}\sum_{j_2=0}^{Q_2-1}\sum_{k_1=1}^{S_1}\sum_{l_2=0}^{S_2}\Pi^{(i_1,j_2,k_1,l_2)} + (\mu_1 p_{12} + \phi_1)\sum_{i_1=1}^{C_2-1}\sum_{j_2=0}^{Q_2-2}\sum_{k_1=1}^{S_1}\sum_{l_2=0}^{S_2}\Pi^{(i_1,j_2,k_1,l_2)}.\end{aligned} \tag{29}$$

5. Steady-State Analysis

The homogeneous Markov process is described with $\{(C_1(t), C_2(t), I_1(t), I_2(t)): t \geq 0\}$ on the state space E, as shown by the structure of the rate matrix $\widetilde{Q}$. As a result, the Markov process limiting the distribution exits.

$$\Pi^{(i_1,j_2,k_1,l_2)} = \lim_{t\longrightarrow\infty} \Pr\left[(C_1(t), C_2(t), I_1(t), I_2(t)) = (i_1, j_2, k_1, l_2)\right]. \tag{30}$$

Let Π, partitioned as $\Pi = (\Pi^{(0)}, \Pi^{(1)}, \Pi^{(2)}, \ldots, \Pi^{(i_1-1)}, \Pi^{(i)}, \ldots)$ represents the steady-state probability vector of $\widetilde{Q}$, where

$$\begin{aligned}\Pi^{(i_1,j_2,k_1)} &= \Pi^{(i_1,j_2,k_1,l_2)}; 0 \leq l_2 \leq S_2,\\ \Pi^{(i_1,j_2)} &= \left(\Pi^{(i_1,j_2,0)}, \Pi^{(i_1,j_2,1)}, \ldots, \Pi^{(i_1,j_2,k_1)}\right); 0 \leq k_1 \leq S_1,\\ \Pi^{(i_1)} &= \left(\Pi^{(i_1,0)}, \Pi^{(i_1,1)}, \ldots, \Pi^{(i_1,j_2)}\right); 0 \leq j_2 \leq C_2,\\ \Pi &= \left(\Pi^{(0)}, \Pi^{(1)}, \ldots, \Pi^{(i_1-1)}, \Pi^{(i_1)}, \ldots\right); 0 \leq i_1 \leq \infty,\end{aligned} \tag{31}$$

where $\Pi^{(i_1,j_2,k_1,l_2)}$ represents the steady state probability for the state (i_1, j_2, k_1, l_2) of the system and Π satisfies the conditions:

$$\Pi\widetilde{Q} = 0 \text{ and } \sum_{i_1=0}^{\infty}\sum_{j_2=0}^{C_2}\sum_{k_1=0}^{S_1}\sum_{l_2=0}^{S_2}\Pi^{(i_1,j_2,k_1,l_2)} = 1. \tag{32}$$

The subvectors satisfy the equations

$$\begin{aligned}&\Pi^0 D_0 + \Pi^1 G_2 = 0\\ &\Pi^0 G_0 + \Pi^1 G_1 + \Pi^2 G_2 = 0\\ &\Pi^1 G_0 + \Pi^2 G_1 + \Pi^3 G_2\\ &\vdots\end{aligned}$$

In general, we can write, $\Pi^{(i_1-1)}G_0$

$$+ \Pi^{(i_1)}G_1 + \Pi^{(i_1+1)}G_2 = 0; \quad i_1 \geq 1. \tag{33}$$

Theorem 1. *When the stability condition (28) exists, the steady state probability vectors Π are obtained using the matrix-geometric technique.*

$$\Pi^{i_1} = \Pi^0 R^{i_1}; i_1 \geq 1, \tag{34}$$

where the matrix R satisfies the matrix quadratic equation and is given by,

$$R^2 G_2 + RG_1 + G_0 = 0. \tag{35}$$

The vector Π^0 can be obtained by solving the equation

$$\Pi^{(0)}\left[D_0 + RG_2\right] = 0, \tag{36}$$

subject to a normalizing condition

$$\begin{aligned}&\sum_{k_1=0}^{\infty}\Pi^{k_1}e = \sum_{k_1=0}^{\infty}\Pi^0 R^{k_1} e = 1\\ &i.e., \Pi^{(0)}(I - R)^{-1}e = 1.\end{aligned} \tag{37}$$

Proof. The theorem follows from the well-known result of matrix-geometric methods ([21]). □

6. System Performance Measures

We can calculate some relevant performance measures for the system after obtaining the steady-state probability vector. In this section, we generate several stationary performance measures of the system to demonstrate the qualitative aspect of the model's outcomes. We can also determine the total expected cost per unit of time by using these parameters.

(1) Expected number of customers in the system is denoted by $E(C)$ and is defined by,

$$E(C) = \sum_{i_1=0}^{\infty} \sum_{j_2=0}^{C_2} \sum_{k_1=0}^{S_1} \sum_{l_2=0}^{S_2} i_1 \Pi^{(i_1,j_2,k_1,l_2)} e + \sum_{i_1=0}^{\infty} \sum_{j_2=0}^{C_2} \sum_{k_1=0}^{S_1} \sum_{l_2=0}^{S_2} j_2 \Pi^{(i_1,j_2,k_1,l_2)} e. \tag{38}$$

(2) Expected inventory level in the system is denoted by $E(I)$ and is given by,

$$E(I) = \sum_{i_1=0}^{\infty} \sum_{j_2=0}^{C_2} \sum_{k_1=1}^{S_1} \sum_{l_2=0}^{S_2} k_1 \Pi^{(i_1,j_2,k_1,l_2)} e + \sum_{i_1=0}^{\infty} \sum_{j_2=0}^{C_2} \sum_{k_1=0}^{S_1} \sum_{l_2=1}^{S_2} l_2 \Pi^{(i_1,j_2,k_1,l_2)} e. \tag{39}$$

(3) Expected Replenishment rate in the system is denoted by $E(R)$ and is given by,

$$E(R) = \theta_1 \sum_{i_1=0}^{\infty} \sum_{j_2=0}^{C_2} \sum_{k_1=0}^{s_1} \sum_{l_2=0}^{S_2} \Pi^{(i_1,j_2,k_1,l_2)} e + \theta_2 \sum_{i_1=0}^{\infty} \sum_{j_2=0}^{N_2} \sum_{k_1=0}^{S_1} \sum_{l_2=0}^{s_2} \Pi^{(i_1,j_2,k_1,l_2)} e. \tag{40}$$

(4) Expected Perishability rate in the system is denoted by $E(P)$ and is given by,

$$E(P) = \beta_1 \sum_{i_1=0}^{\infty} \sum_{j_2=0}^{C_2} \sum_{k_1=1}^{S_1} \sum_{l_2=0}^{S_2} \Pi^{(i_1,j_2,k_1,l_2)} e + \beta_2 \sum_{i_1=0}^{\infty} \sum_{j_2=0}^{C_2} \sum_{k_1=0}^{S_1} \sum_{l_2=1}^{S_2} \Pi^{(i_1,j_2,k_1,l_2)} e. \tag{41}$$

(5) Expected number of departures after completing service in the system is denoted by $E(D)$ and is given by,

$$E(D) = \mu_1 p_{10} \sum_{i_1=1}^{\infty} \sum_{j_2=0}^{C_2} \sum_{k_1=1}^{S_1} \sum_{l_2=0}^{S_2} \Pi^{(i_1,j_2,k_1,l_2)} e + \mu_2 p_{20} \sum_{i_1=0}^{\infty} \sum_{j_2=1}^{C_2} \sum_{k_1=0}^{S_1} \sum_{l_2=1}^{S_2} \Pi^{(i_1,j_2,k_1,l_2)} e. \tag{42}$$

(6) Expected reorder rate in the system is denoted by $E(RO)$ and is defined by,

$$\begin{aligned} E(RO) = {} & (\mu_1 p_{12} + \mu_1 p_{10}) \sum_{i_1=1}^{\infty} \sum_{j_2=0}^{C_2} \sum_{k_1=1}^{s_1} \sum_{l_2=0}^{S_2} \Pi^{(i_1,j_2,k_1+s_1,l_2)} e + (\mu_2 p_{21} + \mu_2 p_{20}) \sum_{i_1=0}^{\infty} \sum_{j_2=1}^{C_2} \sum_{k_1=0}^{S_1} \sum_{l_2=1}^{s_2} \Pi^{(i_1,j_2,k_1,l_2+s_2)} e \\ & + \beta_1 \sum_{i_1=0}^{\infty} \sum_{j_2=0}^{C_2} \sum_{k_1=1}^{s_1} \sum_{l_2=0}^{S_2} \Pi^{(i_1,j_2,k_1+s_1,l_2)} e + \beta_2 \sum_{i_1=0}^{\infty} \sum_{j_2=0}^{C_2} \sum_{k_1=0}^{S_1} \sum_{l_2=1}^{s_2} \Pi^{(i_1,j_2,k_1,l_2+s_2)} e. \end{aligned} \tag{43}$$

(7) Expected reneging rate of the customers in the system is denoted by $E(R_r)$ and is defined by,

$$E(R_r) = \psi_1 \sum_{i_1=1}^{\infty} \sum_{j_2=0}^{C_2} \sum_{k_1=0}^{S_1} \sum_{l_2=0}^{S_2} \Pi^{(i_1,j_2,k_1,l_2)} e + \psi_2 \sum_{i_1=0}^{\infty} \sum_{j_2=1}^{C_2} \sum_{k_1=0}^{S_1} \sum_{l_2=0}^{S_2} \Pi^{(i_1,j_2,k_1,l_2)} e. \quad (44)$$

(8) Expected jockeying rate of the customers in the system is denoted by $E(J_r)$ and is defined by,

$$E(J_r) = \phi_1 \sum_{i_1=1}^{\infty} \sum_{j_2=0}^{C_2} \sum_{k_1=0}^{S_1} \sum_{l_2=0}^{S_2} \Pi^{(i_1,j_2,k_1,l_2)} e + \phi_2 \sum_{i_1=0}^{\infty} \sum_{j_2=1}^{C_2} \sum_{k_1=0}^{S_1} \sum_{l_2=0}^{S_2} \Pi^{(i_1,j_2,k_1,l_2)} e. \quad (45)$$

7. Cost Analysis

The steady-state expected cost rate is numerically analyzed using the following parameters. We define

$$\begin{aligned}
C_w &= \text{waiting cost of a customer per unit time in the system,} \\
C_h &= \text{inventory carrying cost per unit time,} \\
C_{r_1} &= \text{replenishment cost per unit time in the system,} \\
C_p &= \text{perishable cost per unit time in the system,} \\
C_{s_1} &= \text{cost of serving of a customer per unit time in the system,} \\
C_{s_2} &= \text{setup cost of per order,} \\
C_{r_2} &= \text{reneging cost of a customer per unit time in the system,} \\
C_j &= \text{jockeying cost of a customer per unit time in the system.}
\end{aligned} \quad (46)$$

In this model, we introduce the cost function, which is defined as the expected total cost (ETC) of the system, as provided by

$$ETC = C_w E(C) + C_h E(I) + C_{r_1} E(R) + C_p E(P) + C_{s_1} E(D) + C_{s_2} E(RO) + C_{r_2} E(R_r) + C_j E(J_r). \quad (47)$$

It is demanding to illustrate the convexity of the total expected costs because $\Pi' s$ computations are recursive. Although expressing the cost as a function of only two variables is obviously a constrained case and should be avoided, in many situations it may be simple to show the computable nature of the result and the existence of local optima. Normally, a sensitivity analysis as well as some of the most important features of the system may be investigated.

8. Numerical Illustrations

In this section, we provide measures of the system's performance by fixing the parameter values involved in the system. We vary over the parameters $\lambda_1, \lambda_2, \mu_1, \mu_2, \beta_1, \beta_2, \theta_1, \theta_2, \psi_1, \psi_2, \phi_1$, and ϕ_2. The corresponding values of the system performance measures are presented for various values of these parameters. To achieve this, we first fixed the values as follows, $S_1 = S_2 = 3, s_1 = s_2 = 1, Q_1 = Q_2 = 2, C_2 = 2, N = 2, p_{10} = 0.6, p_{20} = 0.5, p_{12} = 0.4, p_{21} = 0.5$. Numerical results are presented in Tables 1–4. We observed the following:

Example 1. We discussed the obtained results which are shown in Table 1 by fixing the values of the parameters $\mu_1 = 3.0, \mu_2 = 2.5, \beta_1 = 0.02;\ \beta_2 = 0.05; \theta_1 = 2.0; \theta_2 = 2.0; \psi_1 = 0.001; \psi_2 = 0.002; \phi_1 = 0.003;\ \phi_2 = 0.005$. The stock outs of items are

Table 1: Impact of arrival rates λ_1 and λ_2 on some performance measures.

(λ_1, λ_2)	$E(C)$	$E(I)$	$E(R)$	$E(P)$	$E(D)$	$E(RO)$	$E(R_r)$	$E(J_r)$
(0.05, 0.03)	0.0571587	2.00245	0.170859	0.0718678	0.144876	0.0648954	0.0000890751	0.000240185
(0.06, 0.04)	0.0703192	2.00126	0.199597	0.0723000	0.181574	0.0677589	0.000110431	0.000298032
(0.07, 0.05)	0.0836815	2.00007	0.228287	0.0727269	0.218332	0.0711782	0.000132239	0.000357098
(0.08, 0.06)	0.0972538	1.99889	0.256932	0.0731485	0.255153	0.0751419	0.000154511	0.000417409
(0.09, 0.07)	0.111045	1.99773	0.285536	0.073565	0.292039	0.0796396	0.000177253	0.000478989
(0.10, 0.08)	0.125062	1.99657	0.314102	0.0739764	0.328992	0.0846617	0.000200477	0.000541864

Table 2: Impact of service rates μ_1 and μ_2 on some performance measures.

(μ_1, μ_2)	$E(C)$	$E(I)$	$E(R)$	$E(P)$	$E(D)$	$E(RO)$	$E(R_r)$	$E(J_r)$
(3.0, 2.5)	0.0571587	2.00245	0.170859	0.0718678	0.144876	0.0648954	0.0000890751	0.000240185
(3.1, 2.6)	0.0553761	2.00258	0.170798	0.0718086	0.144894	0.0647741	0.0000867092	0.000233813
(3.2, 2.7)	0.0537047	2.00268	0.170737	0.0717526	0.14491	0.0646598	0.0000844903	0.000227838
(3.3, 2.8)	0.0521345	2.00277	0.170677	0.0716996	0.144924	0.0645521	0.0000824052	0.000222224
(3.4, 2.9)	0.0506569	2.00284	0.170618	0.0716493	0.144936	0.0644503	0.0000804422	0.00021694
(3.5, 3.0)	0.0492638	2.00289	0.170561	0.0716017	0.144948	0.064354	0.0000785909	0.000211957

Table 3: Impact of reneging rates ψ_1 and ψ_2 on some performance measures.

(ψ_1, ψ_2)	$E(C)$	$E(I)$	$E(R)$	$E(P)$	$E(D)$	$E(RO)$	$E(R_r)$	$E(J_r)$
(0.045, 0.011)	0.0559097	2.00219	0.168914	0.0718337	0.142361	0.0647912	0.001838	0.000235455
(0.040, 0.010)	0.0560488	2.00222	0.16913	0.0718375	0.142641	0.0648028	0.00164306	0.000235982
(0.035, 0.009)	0.0561886	2.00225	0.169347	0.0718413	0.142922	0.0648145	0.00144732	0.000236512
(0.030, 0.008)	0.056329	2.00228	0.169565	0.0718452	0.143204	0.0648262	0.00125076	0.000237044
(0.025, 0.007)	0.0564702	2.00231	0.169784	0.071849	0.143488	0.0648379	0.00105339	0.000237578
(0.020, 0.006)	0.056612	2.00234	0.170005	0.0718529	0.143773	0.0648497	0.000855201	0.000238115

Table 4: Impact of jockeying rates ϕ_1 and ϕ_2 on some performance measures.

(ϕ_1, ϕ_2)	$E(C)$	$E(I)$	$E(R)$	$E(P)$	$E(D)$	$E(RO)$	$E(R_r)$	$E(J_r)$
(0.003, 0.005)	0.0571587	2.002454	0.170859	0.0718678	0.144876	0.0648954	0.0000890751	0.000240185
(0.004, 0.006)	0.0571583	2.002453	0.170863	0.0718681	0.144877	0.0648942	0.0000890792	0.000302238
(0.005, 0.007)	0.0571578	2.002451	0.170868	0.0718684	0.144879	0.064893	0.0000890833	0.0003643
(0.006, 0.008)	0.0571574	2.002450	0.170872	0.0718687	0.14488	0.0648917	0.0000890873	0.000426372
(0.007, 0.009)	0.057157	2.002449	0.170877	0.071869	0.144881	0.0648905	0.0000890914	0.000488454
(0.008, 0.010)	0.0571565	2.002448	0.170881	0.0718694	0.144882	0.0648893	0.0000890955	0.000550545

happening frequently as demand rates λ_1 and λ_2 raise in the system. Thus, the optimal values of performance measures $E(C)$,$E(R)$,$E(P)$,$E(D)$,$E(RO)$,$E(R_r)$, and $E(J_r)$ are increasing and $E(I)$ is decreasing, respectively, because several service facilities are provided to customers in the system.

Example 2. We demonstrated the results of this study shown in Table 2, by considering the values of the parameters $\lambda_1 = 0.05, \lambda_2 = 0.03, \beta_1 = 0.02; \beta_2 = 0.05; \theta_1 = 2.0; \theta_2 = 2.0; \psi_1 = 0.001; \psi_2 = 0.002; \phi_1 = 0.003; \phi_2 = 0.005$. When service rates μ_1 and μ_2 are increases in the system, the optimal values of some performance measures $E(I), E(D)$ are increasing and also $E(C), E(R), E(P), E(RO), E(R_r), E(J_r)$ are decreasing, respectively, as is to be expected. This is because there are service facilities or an adequate number of servers to reduce customer wait times and prevent customers from reneging and jockeying in the system.

Example 3. We examined that according to the results shown in Table 3, by fixing the parameters $\lambda_1 = 0.05, \lambda_2 = 0.03$, $\mu_1 = 3.0$, $\mu_2 = 2.5$, $\beta_1 = 0.02; \beta_2 = 0.05; \theta_1 = 2.0; \theta_2 = 2.0; \phi_1 = 0.003; \phi_2 = 0.005$. As reneging rates ψ_1 and ψ_2 decreases in the system, then the optimal values of performance measures $E(I), E(C), E(R), E(P), E(D)$, $E(RO)$, $E(J_r)$ are increasing, and also $E(R_r)$ is decreasing because of less number of customers may range from the system without service for the long waiting time, which has a great impact on service management systems.

Example 4. We investigated the results depicted in Table 4, by fixing the parameters $\lambda_1 = 0.05, \lambda_2 = 0.03, \mu_1 = 3.0, \mu_2 = 2.5, \beta_1 = 0.02; \beta_2 = 0.05; \theta_1 = 2.0; \theta_2 = 2.0; \psi_1 = 0.001; \psi_2 = 0.002$. If jockeying rates ϕ_1 and ϕ_2 increases in the system, the optimal values of performance measures $E(R), E(P)$, $E(D), E(R_r), E(J_r)$ increases and also $E(C), E(I), E(RO)$,

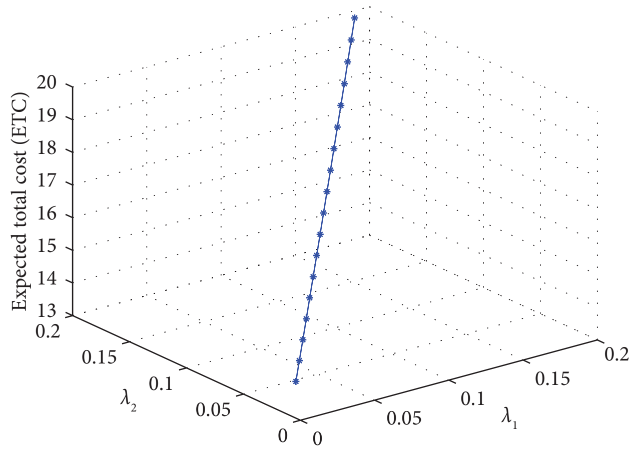

FIGURE 2: Effect of λ_1 and λ_2 on ETC for fix: $\mu_1 = 3.0, \mu_2 = 4.0, \theta_1 = \theta_2 = 2, \beta_1 = 0.03, \beta_2 = 0.04, \psi_1 = 0.001, \psi_2 = 0.002, \phi_1 = 0.002, \phi_2 = 0.003$.

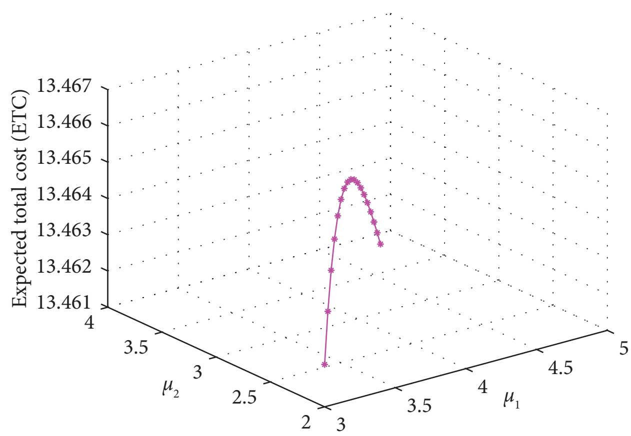

FIGURE 3: Effect of μ_1 and μ_2 on ETC for fix: $\lambda_1 = 0.02, \lambda_2 = 0.03, \theta_1 = \theta_2 = 2, \beta_1 = 0.03, \beta_2 = 0.04, \psi_1 = 0.001, \psi_2 = 0.002, \phi_1 = 0.002, \phi_2 = 0.003$.

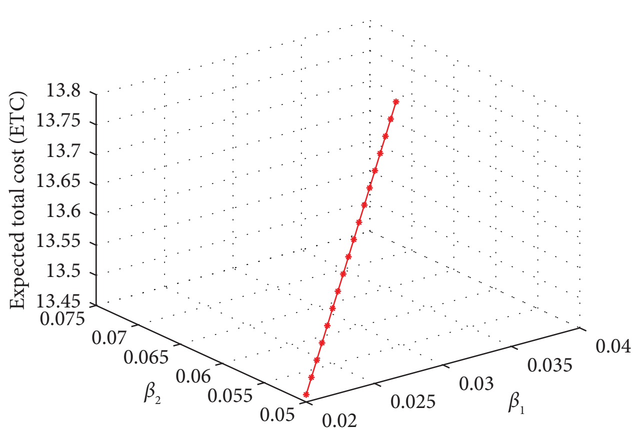

FIGURE 4: Effect of β_1 and β_2 on ETC for fix: $\lambda_1 = 0.02, \lambda_2 = 0.03, \mu_1 = 3.0, \mu_2 = 4.0, \theta_1 = \theta_2 = 2, \psi_1 = 0.001, \psi_2 = 0.002, \phi_1 = 0.002, \phi_2 = 0.003$.

decreases. This happens because, customers switch to a different queue to reduce the waiting time so that they will get faster service.

9. Sensitivity Analysis

In this section, we study the impact of demand rates λ_1 and λ_2, the perishable rates β_1 and β_2, services rates μ_1 and μ_2, lead times θ_1 and θ_2, reneging rates ψ_1 and ψ_2, jockeying rates ϕ_1 and ϕ_2 for server-1 and server-2, respectively, on the total expected cost rate (ETC). Toward this end, we first fix the other parameters' values and costs are $p_{10} = 0.6$; $p_{20} = 0.5$; $p_{12} = 0.4$; $p_{21} = 0.5$; $Q_1 = 2.0$; $Q_2 = 2.0$; $C_w = 5$; $C_h = 6$; $C_{r_1} = 4$; $C_p = 3$; $C_{s_1} = 5$; $C_{s_2} = 2.75$; $C_{r_2} = 3.5$; $C_j = 2.5$. For different values of β_1 and β_2, μ_1 and μ_2, λ_1 and λ_2, θ_1 and θ_2, ψ_1 and ψ_2, ϕ_1 and ϕ_2, graphs of the total expected cost are

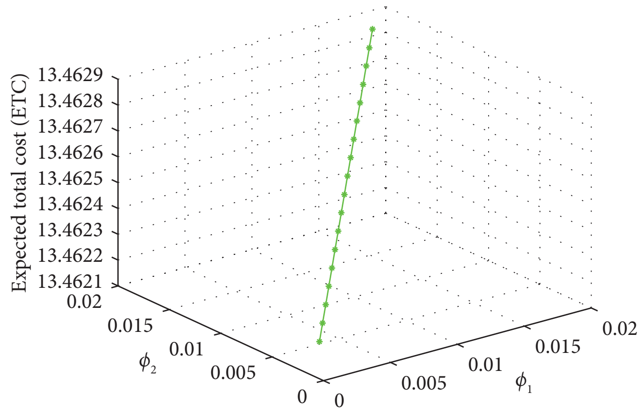

Figure 5: Effect of ϕ_1 and ϕ_2 on ETC: $\lambda_1 = 0.02, \lambda_2 = 0.03, \mu_1 = 3.0, \mu_2 = 4.0, \theta_1 = \theta_2 = 2, \beta_1 = 0.03, \beta_2 = 0.04, \psi_1 = 0.001, \psi_2 = 0.002$.

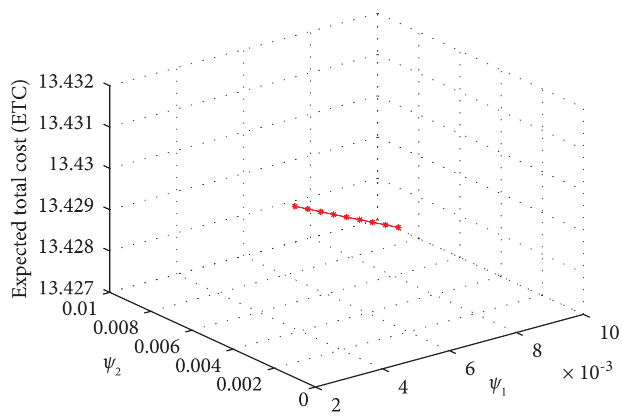

Figure 6: Effect of ψ_1 and ψ_2 on ETC for fix: $\lambda_1 = 0.03, \lambda_2 = 0.02, \mu_1 = 3.0, \mu_2 = 2.0, \theta_1 = \theta_2 = 2, \beta_1 = 0.05, \beta_2 = 0.02, \phi_1 = 0.002, \phi_2 = 0.003$.

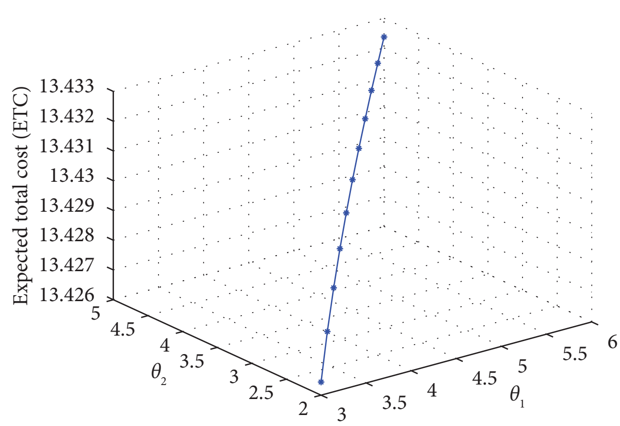

Figure 7: Effect of θ_1 and θ_2 on ETC for fix: $\lambda_1 = 0.03, \lambda_2 = 0.02, \mu_1 = 3.0, \mu_2 = 2.0\beta_1 = 0.05, \beta_2 = 0.02, \psi_1 = 0.0002, \psi_2 = 0.0001, \phi_1 = 0.003, \phi_2 = 0.002$

shown in Figures 2–7. We observed the following figures to exhibit the sensitivity of the system to the effects of varying different parameters:

(i) From Figure 2, we observed that if the arrival rates λ_1 and λ_2 are increased, respectively, the expected total cost (ETC) will increase significantly. This is to be expected because of the higher holding costs of the customers in the system. However, if λ_1 increases and λ_2 decreases, or λ_2 increases and λ_1 decreases, or λ_1 and λ_2 both decreases, then the expected total cost (ETC) is decreased by a significant amount. Therefore, the minimum value of the expected total cost is obtained at the smallest values of λ_1 and λ_2. Hence, ETC is more sensitive on λ_1 to compare λ_2. Similarly, from Figure 3, it is observed that if the service rates μ_1 and μ_2 are increased, respectively, the expected total cost (ETC) will rise significantly over the specified range from $\mu_1 = 3.0$ to $\mu_1 = 3.6$, corresponding to each value of $\mu_2 = 2.0$ to $\mu_2 = 2.6$. However, the expected total cost (ETC) is significantly reduced, over the defined range from $\mu_1 = 3.6$ to $\mu_1 = 4.7$, corresponding to each value of $\mu_2 = 2.6$ to $\mu_2 = 3.7$. So, the global maximum value of the expected total cost is obtained at the values of $\mu_1 = 3.6$ and $\mu_2 = 2.6$. As a result, arrival rates are crucial for the system, which have significant impacts on the expected total cost (ETC) of the optimal values.

(ii) We look to the impact of perishable rates along the defined range of β_1, corresponding to another value of β_2 from Figure 4, if the perishable rates β_1 and β_2 are increased, respectively, then the expected total cost (ETC) is increased. This is because of the higher cost of inventory, which has been damaged frequently in the system. So, the minimum value of the expected total cost occurs at the smallest value of β_1 and β_2. Based on the observations, perishable rates appear to be very sensitive to the system. Additionally, it is noted that if the jockeying rates ϕ_1 and ϕ_2 are raised, the estimated total cost (ETC) will rise significantly over the specified range of ϕ_1 and any value of ϕ_2 from Figure 5. The smallest value of the expected total cost occurs in the least value of ϕ_1 and ϕ_2, which has significant impacts on the anticipated total cost of the optimal solutions. Hence, ETC is more sensitive on ϕ_1 to compare ϕ_2.

(iii) From Figure 6, over the provided range of ψ_1, corresponding to any value of ψ_2 it is observed that if the reneging rates ψ_1 decreases and ψ_2 increases, respectively. Then, the anticipated total cost is significantly reduced. The result is expected due to the lower cost of customers lost from the system. The minimum value of anticipated total cost occurs at the lowest values of ψ_1 and highest value of ψ_2. According to the observation, it looks like the expected total cost is more sensitive on ψ_1 to compare ψ_2 due to reneging of customers from the system. Similarly, in Figure 7 we see that as replenishment rates θ_1 and θ_2 increase, the optimal expected cost rate rises as well. This is to be expected, since, for optimal reorder level and the maximum inventory level of higher order cost is increasing to fulfill the demands. Therefore, the minimum value of the expected total cost (ETC) occurs at the smallest values of θ_1 and θ_2. As a result, expected total cost is more sensitive on θ_1 to compare θ_2.

10. Conclusion

In this paper, we developed a continuous review (s, S) inventory system of a multi-independent $M/M/1$ queue opens the Jackson network with reneging and the jockeying of customers. This model is formulated and manipulated to describe some aspects of reneging and jockeying among customers in a situation based on observation. We used a matrix analytic method to derive the stationary condition of the system as well as the stationary distribution of the combined queue length and on-hand inventory level. Various system performance measures are numerically investigated. We have also presented a sensitivity analysis to show the effect of changes in control variables on the total expected cost function of the system so that, reneging, jockeying, and perishable inventory are maintained to perform the service facility. It is observed that our proposed model can be applied in the various fields of operations management science, production and industrial engineering, telecommunication systems, and other associated areas for practical applications.

Data Availability

All the necessary data used to support the findings of this study are included in the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] Y. Zhang, D. Yue, Li Sun, and J. Zuo, "Analysis of the queueing-inventory system with impatient customers and mixed sales," *Discrete Dynamics in Nature and Society*, vol. 2022, Article ID 2333965, 12 pages, 2022.

[2] A. Krishnamoorthy, D. Shajin, and W. Narayanan, "Inventory with positive service time: a survey," *Advanced Trends in Queueing Theory. Series of Books "Mathematics and Statistics" Sciences*, vol. 2, pp. 201–238.

[3] K. Jeganathan, j. Sumathi, and G. Mahalakshmi, "Markovian inventory model with two parallel queues, jockeying and impatient customers," *Yugoslav Journal of Operations Research*, vol. 26, no. 4, pp. 467–506, 2016.

[4] R. Kumar and S. ., K. Sharma, "A multi-server markovian feedback queue with balking reneging and retention of reneged customers," *AMO-Advanced Modeling and Optimization*, vol. 16, no. 2, pp. 395–406, 2014.

[5] R. Kumar and S. Sharma, "A markovian multi-server queuing model with retention of reneged customers and balking," *International Journal of Operational Research*, vol. 20, no. 4, pp. 427–438, 2013.

[6] A. Choudhury and P. Medhi, "Balking and reneging in multiserver Markovian queuing system," *International Journal of Mathematics in Operational Research*, vol. 3, no. 4, pp. 377–394, 2011.

[7] M. O. Abou-El-Ata and A. M. A. Hariri, "The M/M/c/N queue with balking and renegingM/M/c/NQueue with Balking and Reneging," *Computers & Operations Research*, vol. 19, no. 8, pp. 713–716, 1992.

[8] A. Montazer-Hagighi, J. Medhi, and S. G. Mohanty, "On a multiserver markovian queuing system with balking and reneging," *Computers & Operations Research*, vol. 13, no. 4, pp. 421–425, 1986.

[9] M. E. Islam, M. A. Islam, and S. S. Ahmed, “Base stock stochastic perishable inventory system in Jackson network,” *Journal of Applied Probability and Statistics*, vol. 14, no. 1, pp. 117–136, 2019.

[10] M. A. Islam, M. E. Islam, and A. Rashid, “(s,S) Stochastic inventory system in Jackson network,” *International Journal of Operational Research*, vol. 43, no. 4, pp. 416–436, 2022.

[11] M. Amirul Islam, M. E. Islam, and A. Rashid, “Stochastic optimization of level-dependent perishable inventory system by Jackson network,” *Ain Shams Engineering Journal*, vol. 14, no. 4, Article ID 101935, 2023.

[12] M. Schwarz, C. Wichelhaus, and H. Daduna, “Product form models for queueing networks with an inventory,” *Stochastic Models*, vol. 23, no. 4, pp. 627–663, 2007.

[13] J. R. Jackson, “Networks of waiting lines,” *Operations Research*, vol. 5, no. 4, pp. 518–521, 1957.

[14] A. Krishnamoorthy and N. Anbazhagan, “Perishable inventory system at service facilities with N policy,” *Stochastic Analysis and Applications*, vol. 26, no. 1, pp. 120–135, 2007.

[15] A. Krishnamoorthy, B. Lakshmy, and R. Manikandan, “A survey on inventory models with positive service time,” *Opsearch*, vol. 48, no. 2, pp. 153–169, 2011.

[16] A. Krishnamoorthy, S. S. Nair, and V. C. Narayanan, “An inventory model with server interruptions and retrials,” *Operational Research*, vol. 12, no. 2, pp. 151–171, 2012.

[17] A. Krishnamoorthy, R. Manikandan, and B. Lakshmy, “A revisit to queueing-inventory system with positive service time,” *Annals of Operations Research*, vol. 233, no. 1, pp. 221–236, 2015.

[18] A. Krishnamoorthy and N. C. Viswanath, “Stochastic decomposition in production inventory with service time,” *European Journal of Operational Research*, vol. 228, no. 2, pp. 358–366, 2013.

[19] A. N. Nair, M. J. Jacob, and A. Krishnamoorthy, “The multi server M/M/(s,S) queueing inventory system,” *Annals of Operations Research*, vol. 233, no. 1, pp. 321–333, 2015.

[20] D. Yue, W. Yue, and G. Xu, “Analysis of customers’ impatience in an M/M/1 queue with working vacations,” *Journal of Industrial and Management Optimization*, vol. 8, no. 4, pp. 895–908, 2012.

[21] F. Marcel, *Neuts, Matrix-Geometric Solutions in Stochastic Models: An Algorithmic Approach*, Hopkins University Press, Baltimore, MD, United States, 1981.