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A comprehensive study on geometric shape optical soliton solutions to the time-factional nonlinear Schrödinger-Hirota equation

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ABSTRACT

In this study, we investigate the analytical soliton solutions of a fundamental model, namely the nonlinear Schrödinger-Hirota equation, in the context of beta time-fractional derivative. We adopt the $(\omega'/\omega, 1/\omega)$-expansion method, which is a reliable and straightforward approach to extract fresh and general soliton solutions in terms of hyperbolic, trigonometric, and rational functions. The solitons include anti-kink, anti-bell-shaped, bell-shaped, and periodic solitons. These solitons have significant applications in various scientific fields, such as optical fiber communications, signal processing, plasma physics, and trans-oceanic data transfer. This study demonstrates the significance of fractional-order differentiation in revealing new solitons. We also provide a comprehensive comparison with existing literature in normal and anomalous dispersion regions, highlighting the uniqueness of the solutions. Moreover, the graphical representations are used to illustrate the properties and potential applications of these solitons. This research might contribute to the advancement of nonlinear optical research and technology.

1. Introduction

In the fields of nonlinear physical science and engineering, researchers are extensively studying nonlinear evolution equations (NLEEs). Academics across various research areas consistently investigate different NLEEs that further explain nonlinear phenomena in scientific fields such as optical fibers, nonlinear optics, signal processing, electrodynamics, plasma physics, and other domains.[1-3] Among NLEEs, the nonlinear Schrödinger equation (NLSE) stands out as one of the most appealing equations, attracting the interest of academics. Initially, the NLSE was introduced to describe the movement of quantum particles. Since then, it has expanded to include various phenomena such as polarization, Raman scattering, self-phase modulation, self-stepping, higher dispersion, Kerr-law nonlinearity, and other perturbation terms.[4,5] The nonlinear and linear terms play a significant role in generating wave solutions capable of traveling long distances without losing information.[6,7] While linear terms generally cause pulse dispersion during propagation, nonlinear terms intensify the pulse. Balancing the linear and nonlinear terms led to the generation of a wave solution known as the soliton solution. Solitons are enduring waves that can propagate without altering their shape, energy, or velocity.[8,9] Optical solitons, generated by the NLSE, have numerous applications in various technologies, such as optical fiber communication, satellite communications, network engineering, plasma wave propagation,[10,11] etc. Nowadays, researchers widely exploit different forms of the NLSE to study optical soliton solutions and their applications in light pulse propagation through optical fibers.[12-14]

Over the years, many researchers have studied soliton solutions to examine the characteristics, flaws, and uses of solitons. As a result, several remarkable properties of solitons have been uncovered, such as their extended lifetimes and non-dispersive wave profiles, which maintain a constant size, energy, and velocity. These properties make solitons perfect for transmitting and transferring data in remote areas, providing valuable insights into intricate problems. To examine solitons in NLEEs, researchers have employed different analytical and numerical approaches. For instance, the Kudryashov scheme,[15] the generalized Kudryashov technique,[16,17] the modified extended direct algebraic method,[18,19] the improved F-expansion approach,[20,21] the sine-Gordon expansion procedure,[22,23] the sub-equation mode,[24,25] the modified simple equation method,[26,27] the (G'/G)-expansion scheme,[28,29] the rational (G'/G)-expansion technique,[30,31] the extended generalized (G'/G)-expansion procedure,[32,33] the $(G'/G,1/G)$-expansion process,[34-36]

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the enhanced modified auxiliary equation method,[37,38] etc. are some well-known approaches to examine the soliton solutions of NLEEs.

Beta fractional derivative: Fractional differentiation, the generalization of integer-order differentiation, has revolutionized mathematical modeling and its soliton solutions. Fractional models can be explored analogously to integer-order models, providing more freedom than their integer-order counterparts. Although there are several definitions of fractional derivative in the research sphere, in this study we chose the more reliable beta fractional derivative to develop the fractional model. The beta fractional derivative of a function $f(x, y)$ can be defined as[39]:

$$D_x^\beta f(x,y) = \lim_{\delta\to 0}\frac{f\left(x+h\left(x+\frac{1}{\Gamma\beta}\right)^{1-\beta}, y\right)-f(x,y)}{\delta}, \text{ where } 0<\beta\le 1.$$

The beta fractional derivative adheres to all the fundamental principles followed by the integer-order derivative. We can express the fundamental assumptions of the differentiations in the following form for the beta fractional derivative:

(i) $D_x^\beta f(x,y) = \left(x+\frac{1}{\Gamma\beta}\right)^{1-\beta}\frac{df(x,y)}{dx}$.

(ii) $D_x^\beta(\alpha f(x,y)+\gamma g(x,y)) = \alpha D_x^\beta f(x,y)+\gamma D_x^\beta g(x,y)$.

(iii) $D_x^\beta(f(x,y)g(x,y)) = f(x,y)D_x^\beta g(x,y)+g(x,y)D_x^\beta f(x,y)$.

(iv) $D_x^\beta\left(\frac{f(x,y)}{g(x,y)}\right) = \frac{g(x,y)D_x^\beta f(x,y)-f(x,y)D_x^\beta g(x,y)}{g^2}$.

(v) $D_x^\beta c = 0,\ c$ is a constant.

Therefore, it can be used for any differential equation, while the other fractional derivatives have several disadvantages in terms of implementation.

Governing model: The NLSE plays a crucial role in diverse scientific field especially in the optical fiber pulse propagation, which can be expressed as[40]:

$$iq_t + aq_{xx} + c\left(|q|^{2n}q\right) = 0, \tag{1.1}$$

where the first term describes the temporal evolutions of the spatio-temporal complex wave function $q = q(x, t)$, the parameter a represents the group velocity dispersion coefficient, and the parameter n ($0 < n < 2$) denotes the power law nonlinearity. Selecting $n = 1$, the Eq. (1.1) becomes

$$iq_t + aq_{xx} + c\left(|q|^2 q\right) = 0. \tag{1.2}$$

In Eq. (1.2), $|q|^2$ is the intensity of the wave function. Thus, the parameter c implies the Kerr-law nonlinearity, which is proportional to the intensity of the wave function. It describes the nonlinear response of the medium, where the refractive index of the material is influenced by the intensity of the optical field, leading to effects like self-focusing and self-phase modulation. Introducing third-order dispersion (TOD) and self-steeping terms in Eq. (1.2), the NLSE turns out to be a nonlinear Schrödinger-Hirota equation that holds the following form.[41]

$$iq_t + aq_{xx} + c|q|^2q + i\gamma\left(q_{xxx}+\sigma|q|^2q_x\right) = 0, \tag{1.3}$$

wherein the parameter σ denotes the self-steeping of the wave, and γ signify the third order dispersion.

In the present study, we have taken into account the fractional form of the considered mathematical model so that we can uncover the impact of fractional order along with the accomplishment of the novel solitons. Combining beta fractional derivative, the governing time-fractional Schrödinger-Hirota (SH) equation can be stated as:

$$iD_t^\beta q + aq_{xx} + c|q|^2q + i\gamma\left(q_{xxx}+\sigma|q|^2q_x\right) = 0,\ 0<\beta\le 1. \tag{1.4}$$

In Eq. (1.4), the parameter "a" describes the normal or anomalous group velocity dispersion according to its positive or negative value, respectively. This model is considered to study submarine communications through optical fibers, plasma wave propagations, fluid dynamics, and many other fields relevant to quantum mechanics. Over the years, academics studied this equation to examine the pulse propagation through optics in the presence of Kerr-law nonlinearity, self-steeping, self-phase modulation, and TOD. For instance, Osman et al.,[42] explored bell-shaped and periodic solitons from the SH model via the unified method, Zafar et al.,[41] examined the fractional SH equation with the aid of two well-known approach named the improved (G'/G)-expansion method and the Kudryashov method, Bakodah et al.,[43] analyzed the model through the improve Adomian decomposition scheme, Sharif[44] adopted the modified Jacobi elliptic function approach to inspect the solitons of the generalized SH model, etc.

Though the former literatures explored various soliton solutions to the model, there are still a good number of approaches which had not been adopted to investigate the concerned equation. In this present research, we have taken into consideration a straightforward and reliable scheme, the $(\omega'/\omega, 1/\omega)$-expansion method to investigate the existence of the novel solitons to this model. Furthermore, we have explored various features of solitons, shifting the group velocity dispersion (GVD) region from normal to anomalous.

The rest of this paper is organized as follows: Section 2 presents the key steps required to implement the technique. In Section 3, we provide a detailed discussion of the soliton solutions derived from the equation provided. To further demonstrate the practical applications of these solutions, graphical representations of the solitons are included in Section 4. Section 5 compares our findings to emphasize their novelty. Lastly, Section 6 provides a summary of recent research in the field.

2. Outlines of the method

The $(\omega'/\omega, 1/\omega)$-expansion method is a straightforward approach that is utilized to extract the analytical exact soliton solutions to the NLEEs. This approach uses a second order nonlinear equation, whose general solutions are essential to formulate a series solution to the regarded NLEEs. The second-order nonlinear equation is stated as follows:

$$\omega''(\eta) + \lambda\omega(\eta) = \mu, \tag{2.1}$$

where λ and μ are arbitrary parameters. Let us consider two arbitrary functions as, $\Omega = \omega'/\omega$ and $\chi = 1/\omega$.

Eq. (2.1) yields three individual general solutions depending on the assigned values of λ. The obtained general solutions for different cases have been elucidated in the following section.

Case 1: If we set the value of λ to be negative, the general solution to the Eq. (2.1) are formed by combining hyperbolic functions that holds the subsequent form:

$$\omega(\eta) = \rho_1\sinh\left(\sqrt{-\lambda}\ \eta\right) + \rho_2\cosh\left(\sqrt{-\lambda}\ \eta\right) + \frac{\mu}{\lambda}, \tag{2.2}$$

along with $\chi^2 = -\frac{\lambda\left(\Omega^2-2\chi\mu+\lambda\right)}{\lambda^2\delta+\mu^2}$, $\delta = \rho_1^2-\rho_2^2$, and ρ_1, ρ_2 are arbitrary constants.

Case 2: The value of λ being positive, the general solution of the Eq. (2.1) involves trigonometric functions that can be presented as follows:

$$\omega(\eta) = \rho_1\sin\left(\sqrt{\lambda}\ \eta\right) + \rho_2\cos\left(\sqrt{\lambda}\ \eta\right) + \frac{\mu}{\lambda}, \tag{2.3}$$

with $\chi^2 = \frac{\lambda\left(\Omega^2-2\chi\mu+\lambda\right)}{\lambda^2\delta-\mu^2}$, $\delta = \rho_1^2+\rho_2^2$, and ρ_1, ρ_2 are arbitrary constants.

Case 3: If we assign zero value for the parameter λ, the derived solution to the Eq. (2.1) has the following structure:

$$\omega(\eta) = \frac{\mu}{2}\eta^2 + \rho_1\eta + \rho_2, \tag{2.4}$$

along with $\chi^2 = \frac{1}{\rho_1^2 - 2\mu\rho_2}(\Omega^2 - 2\mu\chi)$.

Now, suppose the governing mathematical model that is to be solved has the following form:

$$P(q,\ q_x, q_t,\ q_{xx},\ D_t^\beta q,\ \ldots) = 0, \tag{2.5}$$

where the polynomial P is comprised of complex wave functions $q = q(x, t)$, and their various differentiations with respect to the spatial and temporal variables. The parameter β signifies the order of beta fractional differentiation with the given constraint $0 < \beta \le 1$. If the given Eq. (2.5) includes imaginary unit i, the considered beta wave transformation becomes

$$q(x,t) = u(\eta)e^{i\theta}, \tag{2.6}$$

together with $\eta = \pm\frac{k_1}{\beta}\left(x + \frac{1}{\Gamma\beta}\right)^\beta \pm \frac{l_1}{\beta}\left(t + \frac{1}{\Gamma\beta}\right)^\beta$, and $\theta = \pm\frac{\alpha_1}{\beta}\left(x_1 + \frac{1}{\Gamma\beta}\right)^\beta \pm \frac{\tau_1}{\beta}\left(t + \frac{1}{\Gamma\beta}\right)^\beta$.

Here, $u(\eta)$ is the amplitude function, and $\theta(x, t)$ is phase function of the wave. The parameters k_1, l_1, τ_1, and α_1 stand for the wave number, velocity, frequency, and phase shift of the wave function, respectively. Exploiting the transformation (2.6) to the regarded model (2.5), it will be altered into an nonlinear ordinary differential equation in terms of new wave variable $u(\eta)$. Let the new attained equation be of the form:

$$L(u,\ u',\ u'', u''',\ u, u^2, \ldots) = 0. \tag{2.7}$$

To evaluate the analytical solutions of Eq. (2.7) using the chosen method, the key steps involved are outlined in the following discussion.

Step 1: In accordance with the assumption of this approach, the exact analytical solutions to Eq. (2.7) are comprised of finite series of two arbitrary functions $\chi(\eta)$, $\Omega(\eta)$, which can be formulated as follows:

$$u(\eta) = \sum_{i=0}^{n} h_i\Omega^i + \sum_{i=1}^{n} r_i\Omega^{i-1}\chi, \tag{2.8}$$

where h_i and r_i are arbitrary parameters, the value of which are to be extracted through the approach. The value of the arbitrary constant n can be derived by using the homogeneous principle of balance between the highest order linear and nonlinear terms included in the equation. The estimated value of n being negative or fractional, the subsequent transformation of the wave function must be followed.

$$u(\eta) = w^{\frac{1}{n}}(\eta).$$

Step 2: Inserting the supposed solution (2.8) in Eq. (2.7), the resultant equation will be formed incorporating the arbitrary functions $\Omega(\eta)$, $\chi(\eta)$, and their differentiations. In consistent with the hypothesis of this scheme, the order of the function $\chi(\eta)$ has to restrained within 1 by replacing the value of χ^2, and the derivatives of $\Omega(\eta)$, $\chi(\eta)$ must be eliminated.

Step 3: Eventually, a system of algebraic equations is to be derived by equating the coefficients of $\Omega^i(\eta)\chi^j(\eta)$ $(i = 0,\ldots n, j = 0, 1)$ on both sides of the resultant equation. Solving this system of equations with the aid of computational software, the value of the required arbitrary parameters incorporated in the conceived trial solution can be derived.

3. Solutions analysis

At the onset, we convert the governing Eq. (1.4) into a nonlinear differential equation after being recognized with the following wave transformation:

$$q(x,t) = u(\zeta)e^{i\theta},$$

together with $\zeta = kx - \frac{v}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)$ and $\theta(x,\ t) = \eta x - \frac{l}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)$, whereas k, v, l, and η characterize the wave number, velocity, frequency, and phase shift of the wave function. Thus, the assessed differential equation is

$$(-a\eta^2 + \gamma\eta^3 + l)u(\zeta) + (c - \gamma\eta\sigma)u(\zeta)^3 - \mathrm{i}(v + k\eta(-2a + 3\gamma\eta))u'(\zeta)$$

$$+\mathrm{i}k\gamma\sigma u(\zeta)^2u'(\zeta) + k^2\left((a - 3\gamma\eta)u''(\zeta) + \mathrm{i}k\gamma u^{(3)}(\zeta)\right) = 0. \tag{3.1}$$

Equating the real and imaginary parts on both sides of Eq. (3.1), two individual equations are generated as follows:

$$(-a\eta^2 + \gamma\eta^3 + l)u(\zeta) + (c - \gamma\eta\sigma)u(\zeta)^3 + k^2(a - 3\gamma\eta)u''(\zeta) = 0, \tag{3.2}$$

and $$(-v + 2ak\eta - 3k\gamma\eta^2)u'(\zeta) + k\gamma\sigma u(\zeta)^2u'(\zeta) + k^3\gamma u^{(3)}(\zeta) = 0. \tag{3.3}$$

Integrating Eq. (3.3) with respect to ζ, we obtain

$$-(v + k\eta(-2a + 3\gamma\eta))u(\zeta) + \frac{1}{3}k\gamma\sigma u(\zeta)^3 + k^3\gamma u''(\zeta) = 0. \tag{3.4}$$

Comparing the coefficients of Eq. (3.2) and Eq. (3.4), the resulted relation is:

$$\frac{-(v + k\eta(-2a + 3\gamma\eta))}{(-a\eta^2 + \gamma\eta^3 + l)} = \frac{k\gamma}{(a - 3\gamma\eta)} = \frac{\frac{1}{3}k\gamma\sigma}{(c - \gamma\eta\sigma)}. \tag{3.5}$$

Solving the Eq. (3.5) for the parameters v and σ, the estimated results are

$$v = \frac{2a^2k\eta - 8ak\gamma\eta^2 + 8k\gamma^2\eta^3 - k\gamma l}{a - 3\gamma\eta},\ \text{and}\ \sigma = \frac{3c}{a}. \tag{3.6}$$

Thus, we have regarded Eq. (3.2) to be solved. To examine soliton solutions, the order of highest-ordered linear and nonlinear terms must be equal, which is known as homogeneous principle of balance. From this hypothesis, the evaluated result for balance variable is, $n = 1$.

Therefore, in accordance with the instruction of the mentioned method, the trail solution of Eq. (3.2) can hold the subsequent mathematical structure:

$$u(\zeta) = h_0 + h_1\Omega(\zeta) + r_1\chi(\zeta). \tag{3.7}$$

Now, substituting the values of $u(\zeta)$, $u'(\zeta)$, $u''(\zeta)$, v, and σ in Eq. (3.2) and executing the described key steps of the method, the subsequent solution sets for three distinct cases have been resulted.

Category 1. (λ < 0): This condition on λ leads to develop three sets of solutions as asserted below:

$$\text{Set}-1:\ h_0 = 0,\ h_1 = \pm\frac{\sqrt{-\lambda}r_1}{\sqrt{\delta\lambda^2 + \mu^2}},\ k = \pm\frac{\sqrt{2}\sqrt{-c}\sqrt{-\lambda}r_1}{\sqrt{a\delta\lambda^2 + a\mu^2}},\ l = \eta^2(a - \gamma\eta) - \frac{c(a - 3\gamma\eta)\lambda^2r_1^2}{a(\delta\lambda^2 + \mu^2)},\ \sigma = \frac{3c}{a},$$

$$v = \frac{2k\eta(a - 2\gamma\eta)^2 - k\gamma l}{a - 3\gamma\eta}.$$

$$\text{Set}-2:\ h_0 = 0,\ h_1 = 0,\ k = \pm\frac{\sqrt{-c}r_1}{\sqrt{2}\sqrt{a}\sqrt{\delta}\sqrt{-\lambda}},\ l = \eta^2(a - \gamma\eta) + \frac{c(a - 3\gamma\eta)r_1^2}{2a\delta},\ \mu = 0,\ \sigma = \frac{3c}{a},$$

$$v = \frac{2k\eta(a-2\gamma\eta)^2 - k\gamma l}{a-3\gamma\eta}.$$

$$\text{Set}-3: \; h_0 = 0, \; h_1 = \pm\frac{\sqrt{2}\sqrt{a}k}{\sqrt{-c}}, \; r_1 = 0, \; \omega = a\eta^2 - \gamma\eta^3 - 2ak^2\lambda + 6k^2\gamma\eta\lambda, \; \mu = 0, \; \sigma = \frac{3c}{a},$$

$$v = \frac{2k\eta(a-2\gamma\eta)^2 - k\gamma l}{a-3\gamma\eta}.$$

Inserting the assessed values of parameters of Set-1, the general solutions to Eq. (3.2) can be structured from Eq. (3.7) that holds the following form:

$$u(\zeta) = \pm\frac{r_1}{\frac{\mu}{\lambda} + \sinh(\zeta\sqrt{-\lambda})g_1 + \cosh(\zeta\sqrt{-\lambda})g_2}\left\{1 \pm\frac{\lambda(\cosh(\zeta\sqrt{-\lambda})g_1 + \sinh(\zeta\sqrt{-\lambda})g_2)}{\sqrt{\delta\lambda^2+\mu^2}}\right\}, \tag{3.8}$$

where the functions $\Omega(\zeta)$ and $\chi(\zeta)$ consist of hyperbolic functions for the considered condition $\lambda < 0$. Replacing the value of $u(\zeta)$ in Eq. (3.1), the required exact soliton solution to Eq. (1.4) can be formulated as:

$$q(x,t) = \pm\frac{e^{i\theta}r_1}{\frac{\mu}{\lambda} + \sinh(\zeta\sqrt{-\lambda})g_1 + \cosh(\zeta\sqrt{-\lambda})g_2}\left\{1 \pm\frac{\lambda(\cosh(\zeta\sqrt{-\lambda})g_1 + \sinh(\zeta\sqrt{-\lambda})g_2)}{\sqrt{\delta\lambda^2+\mu^2}}\right\}, \tag{3.9}$$

including $\zeta = kx - \frac{v}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)$, $\theta(x,t) = \eta x - \frac{l}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)$, and $\delta = g_1^2 - g_2^2$. Solution (3.9) contains several free parameters g_1, g_2, μ, and λ. As the solution (3.9) is derived for the condition $\lambda < 0$, we cannot assign zero value for λ. However, g_1, g_2, and μ can be associated with the null value. If we set $g_1 = 0$ and $\mu = 0$, the obtained result from solution (3.9) is

$$q(x,t) = \pm\frac{e^{i\theta}r_1}{g_2}\left\{\operatorname{sech}\left(\zeta\sqrt{-\lambda}\right) \pm i\tanh\left(\zeta\sqrt{-\lambda}\right)\right\}. \tag{3.10}$$

Similarly, for g_2, $\mu = 0$, the solution (3.9) can be altered to the following form:

$$q(x,t) = \pm\frac{e^{i\theta}r_1}{g_1}\left\{\operatorname{csch}\left(\zeta\sqrt{-\lambda}\right) \pm \coth\left(\zeta\sqrt{-\lambda}\right)\right\}. \tag{3.11}$$

Analogously, the remaining solution sets have been utilized to construct the exact soliton solutions to Eq. (1.4), which are expressed as

$$q(x,t) = \frac{r_1 e^{i\theta}}{\sinh(\zeta\sqrt{-\lambda})g_1 + \cosh(\zeta\sqrt{-\lambda})g_2}. \tag{3.12}$$

$$q(x,t) = \pm\frac{e^{i\theta}\sqrt{2}\sqrt{a}k\sqrt{-\lambda}(\cosh(\zeta\sqrt{-\lambda})g_1 + \sinh(\zeta\sqrt{-\lambda})g_2)}{\sqrt{-c}(\sinh(\zeta\sqrt{-\lambda})g_1 + \cosh(\zeta\sqrt{-\lambda})g_2)}, \tag{3.13}$$

together with $\zeta = kx - \frac{v}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)$, and $\theta(x,t) = \eta x - \frac{l}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)$. Similarly, solutions (3.12) and (3.13) can be transformed to other solutions in terms of elementary functions by selecting zero value for the parameters μ, g_2, and g_1.

Category 2. ($\lambda > 0$): Considering this constraint of λ and executing the steps of the mentioned approach affiliated with this case, three sets of solutions for the chosen arbitrary parameters have been resulted. The evaluated solution sets are as follows:

$$\text{Family } 1: \; h_0 = 0, \; h_1 = \pm\frac{\sqrt{\lambda}r_1}{\sqrt{\delta\lambda^2-\mu^2}}, \; k = \pm\frac{\sqrt{2}\sqrt{c}\sqrt{\lambda}r_1}{\sqrt{-a\delta\lambda^2+a\mu^2}}, \; l = \eta^2(a-\gamma\eta) + \frac{c(a-3\gamma\eta)\lambda^2 r_1^2}{a(\delta\lambda^2-\mu^2)}, \; \sigma = \frac{3c}{a},$$

$$v = \frac{2k\eta(a-2\gamma\eta)^2 - k\gamma l}{a-3\gamma\eta}.$$

$$\text{Family } 2: \; h_0 = 0, \; h_1 = 0, \; k = \pm\frac{\sqrt{c}r_1}{\sqrt{2}\sqrt{-a}\sqrt{\delta}\sqrt{\lambda}}, \; l = \eta^2(a-\gamma\eta) - \frac{c(a-3\gamma\eta)r_1^2}{2a\delta}, \; \mu = 0, \; \sigma = \frac{3c}{a},$$

$$v = \frac{2k\eta(a-2\gamma\eta)^2 - k\gamma l}{a-3\gamma\eta}.$$

$$\text{Family } 3: \; h_0 = 0, \; h_1 = \pm\frac{\sqrt{2}\sqrt{-a}k}{\sqrt{c}}, \; r_1 = 0, \; l = a\eta^2 - \gamma\eta^3 - 2ak^2\lambda + 6k^2\gamma\eta\lambda, \; \mu = 0, \; \sigma = \frac{3c}{a}, \; v = \frac{2k\eta(a-2\gamma\eta)^2 - k\gamma l}{a-3\gamma\eta}.$$

Now, substituting the obtained values of the parameters and the functions $\Omega(\zeta)$, $\chi(\zeta)$ from the aforementioned solution Family 1 to (3.7), the generated solutions for Eq. (3.2) is

$$u(\zeta) = \pm\frac{r_1}{\frac{\mu}{\lambda} + \sin(\zeta\sqrt{\lambda})g_1 + \cos(\zeta\sqrt{\lambda})g_2}\left\{1 \pm\frac{\lambda(\cos(\zeta\sqrt{\lambda})g_1 - \sin(\zeta\sqrt{\lambda})g_2)}{\sqrt{\delta\lambda^2-\mu^2}}\right\}. \tag{3.14}$$

To obtain the exact soliton solution to the regarded Eq. (1.4), we have placed the value of $u(\zeta)$ in Eq. (3.1). Thus, the resultant solution is as follows:

$$q(x,t) = \pm\frac{e^{i\theta}r_1}{\frac{\mu}{\lambda} + \sin(\zeta\sqrt{\lambda})g_1 + \cos(\zeta\sqrt{\lambda})g_2}\left\{1 \pm\frac{\lambda(\cos(\zeta\sqrt{\lambda})g_1 - \sin(\zeta\sqrt{\lambda})g_2)}{\sqrt{\delta\lambda^2-\mu^2}}\right\}, \tag{3.15}$$

with $\zeta = kx - \frac{v}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)$, $\theta(x,t) = \eta x - \frac{l}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)$, and $\delta = g_1^2 + g_2^2$.

In solution (3.15), g_1, g_2, μ, and λ are arbitrary parameters. Assigning null value for g_1 and μ, the solution (3.15) modifies to a new form as expressed below:

$$q(x,t) = \pm\frac{e^{i\theta}r_1}{g_2}\left\{\sec\left(\zeta\sqrt{\lambda} \pm \tan\left(\zeta\sqrt{\lambda}\right)\right\}. \tag{3.16}$$

Similarly, another form of solution can be ascertained from solution (3.15) by selecting zero value for g_2, and μ. However, the remaining solution sets have been put into use to develop the solutions to the Eq. (1.4) that hold the corresponding forms as given below:

$$q(x,t) = \frac{e^{i\theta}r_1}{\sin(\zeta\sqrt{\lambda})g_1 + \cos(\zeta\sqrt{\lambda})g_2}. \tag{3.17}$$

$$q(x,t) = -\frac{e^{i\theta}\sqrt{2}\sqrt{a}k\sqrt{\lambda}(\cos(\zeta\sqrt{\lambda})g_1 - \sin(\zeta\sqrt{\lambda})g_2)}{\sqrt{-c}(\sin(\zeta\sqrt{\lambda})g_1 + \cos(\zeta\sqrt{\lambda})g_2)}, \tag{3.18}$$

where $\zeta = kx - \frac{v}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)$, $\theta(x,t) = \eta x - \frac{l}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)$, and $\delta = g_1^2 + g_2^2$.

Category 3. ($\lambda = 0$): The execution of the method for this considered condition have yielded three sets of solutions for the presumed parameters in trial solution, which are stated as:

$$\text{Family 1}: \ h_0 = 0, \ = -\frac{\sqrt{a}k}{\sqrt{2}\sqrt{-c}}, \ r_1 = \frac{\sqrt{a}k\sqrt{g_1^2 - 2\mu g_2}}{\sqrt{2}\sqrt{-c}}, \ l = -\eta^2(-a+\gamma\eta), \ \sigma = \frac{3c}{a}, \ \nu = \frac{2k\eta(a-2\gamma\eta)^2 - k\gamma l}{a - 3\gamma\eta}.$$

$$\text{Family 2}: \ h_0 = 0, \ h_1 = 0, \ k = \pm\frac{\sqrt{-c}r_1}{\sqrt{2}\sqrt{a}g_1}, \ l = -\eta^2(-a+\gamma\eta), \ \mu = 0, \ \sigma = \frac{3c}{a}, \ \nu = \frac{2k\eta(a-2\gamma\eta)^2 - k\gamma l}{a - 3\gamma\eta}.$$

$$\text{Family 3}: \ h_0 = 0, \ h_1 = \pm\frac{\sqrt{2}\sqrt{a}k}{\sqrt{-c}}, \ r_1 = 0, \ l = -\eta^2(-a+\gamma\eta), \ \mu = 0, \ \sigma = \frac{3c}{a}, \ \nu = \frac{2k\eta(a-2\gamma\eta)^2 - k\gamma l}{a - 3\gamma\eta}.$$

The estimated values of the parameters are to be utilized in (3.7) to form the general solution to Eq. (3.2). Acknowledging the solution Set 1, the evaluated general solution for the Eq. (3.2) can be formulated as

$$u(\zeta) = \frac{\sqrt{a}k}{\sqrt{2}\sqrt{-c}\left(\frac{\zeta^2\mu}{2} + \zeta g_1 + g_2\right)}\left\{(\zeta\mu + g_1) - \sqrt{g_1^2 - 2\mu g_2}\right\}. \tag{3.19}$$

Now, the exact soliton solutions to the Eq. (1.4) have been generated by using the value of $u(\zeta)$ in the Eq. (3.1). The mathematical expression of the calculated solution is

$$q(x,t) = \frac{e^{i\theta}\sqrt{a}k}{\sqrt{2}\sqrt{-c}\left(\frac{\zeta^2\mu}{2} + \zeta g_1 + g_2\right)}\left\{(\zeta\mu + g_1) - \sqrt{g_1^2 - 2\mu g_2}\right\}. \tag{3.20}$$

In a similar way, the derived solution for the solution Set 2 can be asserted as

$$q(x,t) = \frac{e^{i\theta}r_1}{\zeta g_1 + g_2}. \tag{3.21}$$

Whereas, Set 3 has delivered the following solution to Eq. (1.4)

$$q(x,t) = -\frac{e^{i\theta}\sqrt{2}\sqrt{a}kg_1}{\sqrt{-c}(\zeta g_1 + g_2)}, \tag{3.22}$$

along with $\zeta = kx - \frac{\nu}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)$, and $\theta(x,t) = \eta x - \frac{l}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)$.

4. Analysis of the results with graphical representations

In this section, we have discussed the characteristic analysis of solutions obtained through graphical representation and their application in scientific fields. Graphical representations include three- and two-dimensional graphs and contour plots of solutions within the range $0 \le x \le 20$, and $0 \le t \le 20$.

The modulus of solution (3.9) presents an anti-kink soliton for the chosen parametric values $\beta = 0.99$, $c = -6.52$, $a = 4.25$, $\lambda = -2.485$, $g_1 = 12.15$, $g_2 = 1.24$, $r_1 = 2.8$, $\eta = 2.15$, $\mu = -11.25$, $\gamma = -14.35$ and presented in Fig. 1. It is noticed that anti-kink solitons hold a long-lasting life span and experience a shock while propagating with time. It is a stable and firm solitary wave solutions, which does not change its shape, energy, velocity, and other properties even after the collision with other soliton. These noteworthy characteristics are crucial for data transmission, signal processing, communication systems, and to study many other fields. The kink soliton is acquainted as the dark soliton in optical fiber which occurs in normal group velocity dispersion (GVD) region, where the value of a (group velocity dispersion) is positive. The two-dimensional graph infers the stability of the anti-kink soliton, which shows that the energy of the anti-kink soliton remains unaltered with the varying values of β. Though the tough of the anti-kink soliton takes place at different points, it does not loss its stability.

Now, changing the value of $a = -11.55$, $r = 0.1$, $\eta = 1.35$, $g_1 = 5$, and $g_2 = -5$, the modulus of solution (3.9) exhibits a periodic soliton as shown in Fig. 2. It is known as periodic soliton that takes place in the anomalous GVD region. Like the kink soliton, periodic soliton is also a stable, nondispersive, and long-lasting soliton. The stability nature is crucial to persistent the collision with other solitons. Periodic solitons are used to transfer data, information, and signals over a long-distance region. Its periodic nature makes it suitable to study the fusion research, plasma waves, photonic crystals, and others.

Fig. 3 shows the physical features of the modulus of solution (3.12), which is drawn for the values $\beta = 0.99$, $c = -2.02$, $a = -9.65$, $\lambda = -4.355$, $g_1 = -11.75$, $g_2 = -6.3$, $r_1 = -8$, $\eta = 0.8$, $\gamma = -5.85$. The modulus of solution (3.12) exhibits a periodic soliton for the suitable values of parameters. Fig. 3(b) portrait the impact of the fractional order differentiation on the characteristics of solitons. It can be concluded that the wavelength is broadened with the decreasing values of β. Consequently, the energy of the soliton is also diminished that leads the soliton to loss its stability. Therefore, regulating the value of fractional-order, we can adjust the energy of the periodic soliton.

Assigning a positive value to the parameter a or c (normal dispersion region), the modulus of solution (3.12) turns to a singular soliton. To examine the nature of singularities and discover the way to remove singularities, singular solitons play vital roles in mathematics. Singular solitons have great importance in mathematical inspections, even though it has no physical applications.

The modulus of solution (3.15) depicts an anti-bell-shaped soliton in the positive region of GVD or normal dispersion region, which turns to a

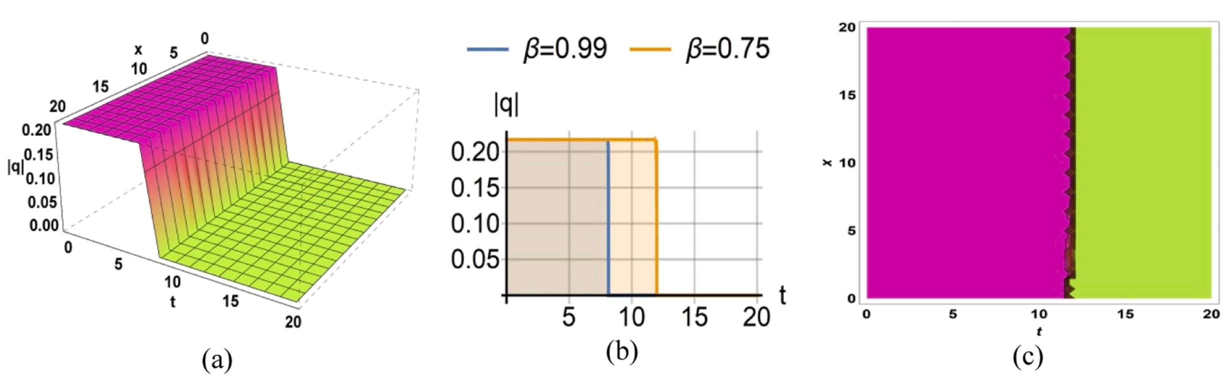


Fig. 1. Graph of modulus of solution of (3.9) in normal dispersion region.

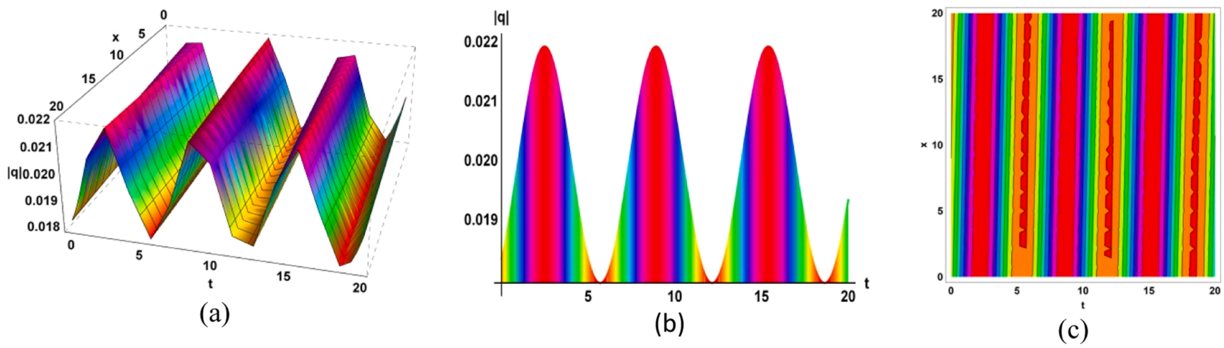


Fig. 2. Graph of modulus of solution of (3.9) in anomalous dispersion region.

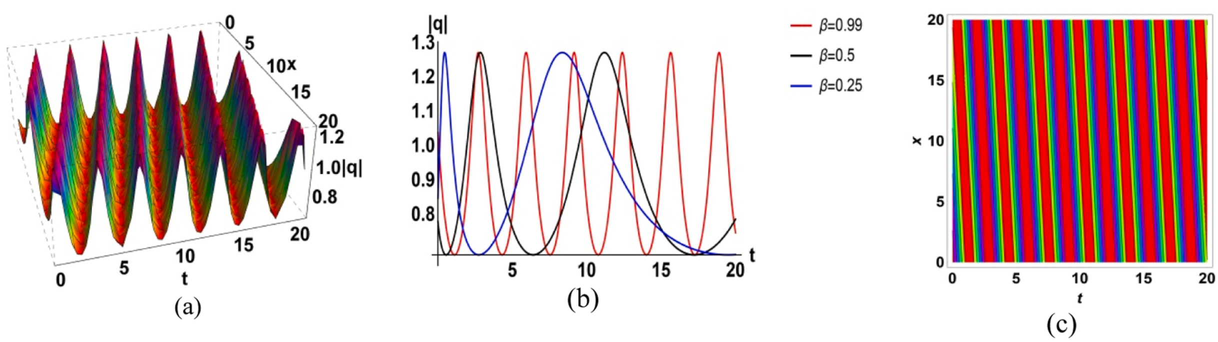


Fig. 3. Graph of modulus of solution of (3.12) in normal dispersion region.

periodic soliton for negative GVD or anomalous GVD region. Fig. 4 represents the graphical illustrations of the modulus of solution (3.15) that has been drawn for the values $\beta = 0.99$, $c = -10$, $\lambda = 0.1$, $g_1 = 4.45$, $g_2 = -9.56$, $r_1 = -8.2$, $\eta = 0.3$, $\mu = -7.9$, $\gamma = -0.55$.

Anti-bell-shaped soliton has a trough in the center of the wave profile shown in Fig. 5. It can maintains its shape and amplitude over long distances without spreading out, making them suitable for information transmission.

Fig. 5(a) visualizes the 2D graphs of the solution (3.15) for different times, whereas Fig. 5(b) illustrates the 2D graphs of solution (3.15) for different fractional order. From these figures, we can observe that the soliton solution can maintain its shape while propagating with time, but its stability can be perturbed by changing the value of β (beta fractional)

Fig. 6 is portrayed by choosing the values $\beta = 0.99$, $c = -3.82$, $a = -9.05$, $\lambda = 0.1$, $g_1 = -15$, $g_2 = -0.7$, $r_1 = -1.25$, $\eta = -0.35$, $\gamma = 14.95$, which presents a bell-shaped soliton for the modulus of solution (3.17). Bell-shaped solitons are characterized by a localized and high intensity profile. The shape resembles a bell curve, with the maximum intensity occurring at the center and diminishing towards the edges. Bell solitons are robust against external perturbations. They can withstand disturbances and maintain their integrity, contributing to their reliability in various applications. Their ability to propagate over long distances without distortion makes them valuable for transmitting information in fiber optic networks

5. Comparison

This section is set out to interpret the novelty of the results provided by this study. We have made a comparison of the results provided in this study with the existing results documented in prior literatures. The mentioned comparison is presented in Table 1 as given below context:

Table 1 clearly demonstrates that this study provides new solitons with significant applications in various scientific research areas and technological developments. Furthermore, the comparison of the results confirms that the adopted approach is effective in investigating the NLEEs. The new solitons provides a straightforward way to understand the complex behavior related to the considered model.

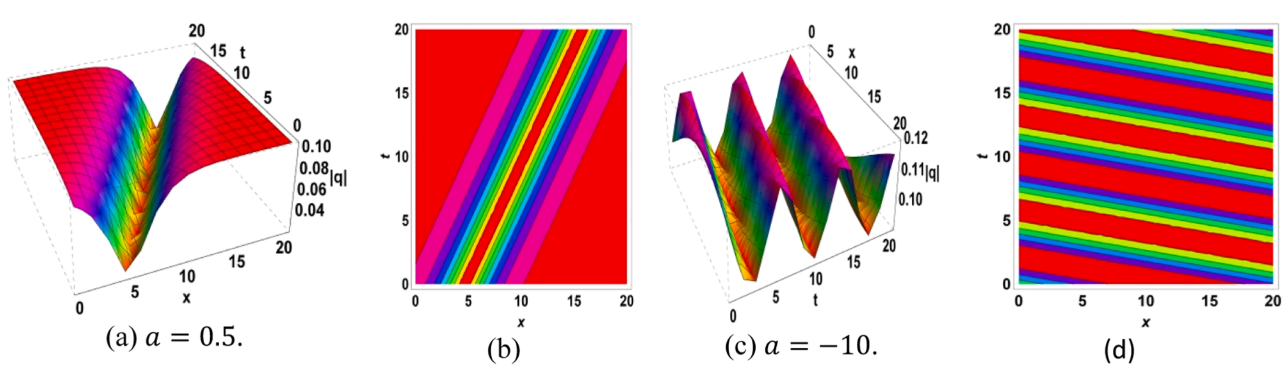
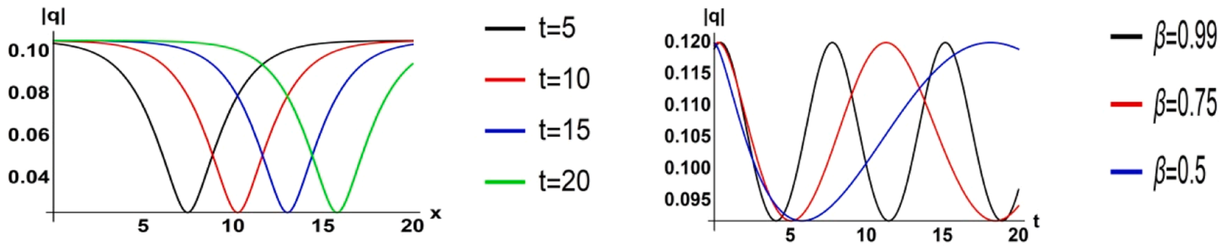
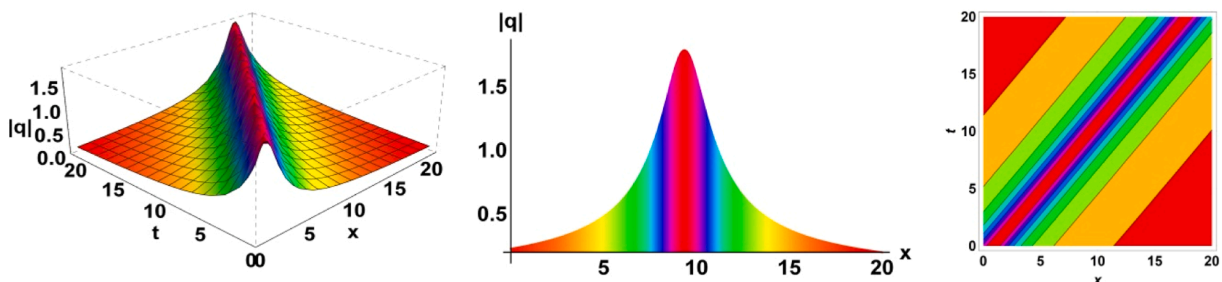


Fig. 4. Graph of modulus of solution of (3.15).



(a) $a = 0.5$. (b) $a = -10$.

Fig. 5. 2D Graphs of modulus of solution of (3.15).



(a) (b) (c)

Fig. 6. Graphs of modulus of solution of (3.17).

Table 1
Comparison of the findings.

Study on this equation	What type of fractional derivative is applied?	Resulted solitons on the nonlinear SH equation			
		Anti-kink soliton	Bell-shaped soliton	Anti-bell-shaped soliton	Periodic soliton
Present study	Beta fractional	√	√	√	√
Zafar et al.[41]	Conformable	×	√	×	√
Sharif, A.[44]	Conformable	×	×	×	√
Osman et al.[42]	×	×	√	×	√

6. Conclusion

In this study, we successfully established fresh and generic soliton solutions to the time-fractional nonlinear Schrödinger-Hirota equation using the $(\omega'/\omega, 1/\omega)$-expansion approach. This research highlights the significance of fractional calculus in unveiling the impact of fractional derivatives on solitons, such as anti-kink, bell-shaped, anti-bell-shaped, and periodic solitons. These solitons have considerable potential for applications in various scientific fields, including optical fiber communications, plasma physics, and data transmission technologies. The graphical representations provided further demonstrate the stability and practical utility of these solitons under different conditions. This work extends the understanding of soliton dynamics in nonlinear evolution equations and overlays the way for future studies exploring the complexities of such models using fractional derivatives. The findings of this research are expected to contribute significantly to advancements in nonlinear optical research and technology.

Ethical approval

This article does not involve with any human or animal studies. We also confirm that, we have read and abided by the statement of ethical standards for the manuscript submission to this journal and that the manuscript has not been copyrighted, published, or submitted elsewhere.

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Availability of data and materials

The data used to support the findings of this study are included within the article.

CRediT authorship contribution statement

Mst. Munny Khatun: Conceptualization, Resources, Software, Methodology, Visualization, Writing – original draft. **Shahansha Khan:** Investigation, Validation, Data curation, Writing – review & editing. **M. Ali Akbar:** Data curation, Formal analysis, Project administration, Funding acquisition, Supervision, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no competing interests. All the authors with the consultation of each other completed this research and drafted the manuscript together. All authors have read and approved the final manuscript.

Data availability

No data was used for the research described in the article.

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