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ORIGINAL PAPER

Modified Double Sub-equation Method for Finding Complexiton Solutions to the (1 + 1) Dimensional Nonlinear Evolution Equations

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Abstract This paper reflects the execution of a reliable technique which we proposed as a modified double sub-equation method for solving nonlinear evolution equations. The proposed scheme has been successfully hardened on a very important evolution equation namely the (1 + 1)-dimensional Burger equation and the (1 + 1)-dimensional Gardner equation (or combined KdV–mKdV). As a results, we found more complexiton solutions in-terms of trigonometric, hyperbolic functions. Finally, the interaction phenomena of the achieved complexiton solutions between solitary waves and/or periodic waves are presented with in depth derivation. The outcome can be used to enhance the dynamical activities of higher dimensional nonlinear wave field areas.

Keywords The modified double-sub equation method · The (1 + 1)-dimensional Burger equation · Traveling wave solutions · Complexiton solutions

Mathematics Subject Classification 35C07 · 35C08 · 35P99

Introduction

Nonlinear evolution equations (NLEEs) play a noteworthy role in various scientific and engineering fields such as applied mathematics, plasma physics, fluid dynamics, optical fibers, biology, solid state physics, chemical physics, mechanics and geochemistry. In recent years, both mathematicians and physicists have devoted considerable effort to study of soliton solutions of nonlinear partial differential equations (PDEs) and a number of powerful methods were presented. For instance the inverse scattering theory [1], Darboux transformation [2],

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the Hirota's bilinear method [3], the sech-function method [4,5], the homogeneous balance method [6], Bäcklund transformation method [7], the hyperbolic tangent function series method [8,9], the sine–cosine method [10], the (G'/G)-expansion method [11,12], the multiple exp-function method [13], the Jacobi elliptic function expansion method [14,15]. These algebraic methods have the power to give a clear picture of the relation between different terms of nonlinear wave equations and are to simplify the routine calculation of the method. One of the most effectively straightforward methods to constructing exact solutions of PDEs is the sub-equation method [16–19]. The complexiton solution, firstly introduced by Ma [20], can be constructed by the multiple Riccati equations rational expansion method [21], which make use of two Riccati equations with the same variable. Chen et al. [22] has presented the double sub-equation method using two ordinary differential equations (ODEs) with different independent variables. Complexiton solutions are obtained by combining elementary functions and the Jacobi elliptic functions using double sub-equation method [22]. Burger equation [23,24] and Gardner equation [26,27] are solved by many researcher for finding complexiton solutions. On the other hand, Burgers equation with space-and time-fractional order and and time-fractional Boussinesq–Burger's equations [30–32] are solved for soliton solutions which arise in propagation of shallow water waves.

In this paper, the modified double sub-equation method is proposed for constructing complexiton solutions of nonlinear partial differential equations (PDEs). We apply this method to the Burger's equation [24,25] and to the Gardner equation [26,27]. We will generate many new types of complexiton solutions combining elementary functions and the Jacobi elliptic functions. It makes the modified double sub-equation method more thoroughly.

Materials and Methods

In the following, we described the main steps of modified double sub-equation method.

Step 1 Consider a nonlinear partial differential equation (NLPDE), say in two independent variables x and t, is given by

$$\Re(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \ldots) = 0 \tag{1}$$

where $u = u(x,t)$ is an unknown function, $\Re$ is a polynomial of $u = u(x,t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved.

Step 2 For the suggested method, we assume that the solutions of Eq. (1) are as follows:

$$u(x,t) = a_0 + \frac{a_1\varphi(\xi) + a_2\psi(\eta)}{\lambda_0 + \lambda_1\varphi(\xi)\psi(\eta)} \tag{2}$$

where $a_0 = a_0(x,t)$, $a_1 = a_1(x,t)$, $a_2 = a_2(x,t)$, $\xi = \xi(x,t)$, $\eta = \eta(x,t)$ are all functions of x and t, λ_0 and λ_1 are arbitrary nonzero constants to be determined later. The new functions $\varphi(\xi)$ and $\psi(\eta)$ satisfy

$$\varphi'(\xi) = \frac{d\varphi(\xi)}{d\xi} = q_1 + p_1\varphi^2(\xi) \tag{3}$$

$$\text{and } \psi'(\eta) = \frac{d\psi(\eta)}{d\eta} = q_2 + p_2\psi^2(\eta), \tag{4}$$

where $\xi = k_1x + w_1t$ and $\eta = k_2x + w_2t$ respectively, which are known as wave transformation of Eq. (1).

Step 3 The general solutions of the Riccati Eqs. (3, 4) [21] are as follows:

$$\frac{d\varphi(\xi)}{d\xi} = q_1 + p_1\varphi^2(\xi)$$

i. When $q_1 = 1, p_1 = -1$,

$$\varphi(\xi) = \tanh(\xi), \varphi(\xi) = \coth(\xi), \tag{5}$$

ii. When $q_1 = p_1 = \pm\frac{1}{2}$,

$$\varphi(\xi) = \sec(\xi) \pm \tan(\xi), \varphi(\xi) = \csc(\xi) \pm \cot(\xi), \tag{6}$$

iii. When $q_1 = p_1 = 1$,

$$\varphi(\xi) = \tan(\xi), \tag{7}$$

iv. When $q_1 = p_1 = -1$,

$$\varphi(\xi) = \cot(\xi) \tag{8}$$

v. When $q_1 = \frac{1}{2}, p_1 = -\frac{1}{2}$,

$$\varphi(\xi) = \tanh(\xi) \pm i\sec h(\xi), \varphi(\xi) = \coth(\xi) \pm \csc h(\xi), \tag{9}$$

vi. When $q_1 = 0, p_1 = 1$,

$$\varphi(\xi) = -\frac{1}{p_1\xi + w}, \tag{10}$$

Step 4 By substituting Eq. (2) into Eq. (1) along with Eqs. (3) and (4) yields a system of equations with respect to $\varphi^m\psi^n$, $(m = 0, 1, 2, \ldots, n = 0, 1, 2, \ldots)$ then set all coefficients of $\varphi^m\psi^n$ in the obtained system of equations to be zero, we obtain a set of over-determined PDEs with respect to $a_0, a_1, a_2, k_1, w_1, k_2, w_2, \lambda_0$ and λ_1.

By solving the over-determined PDEs by use of symbolic computation system Maple, we end up the explicit expressions for $a_0, a_1, a_2, k_1, w_1, k_2, w_2, \lambda_0, \lambda_1$. Using the results obtained in the above steps and the various solutions of Eqs. (3, 4), we can derive many solutions for Eq. (1).

Advantages: The main advantages of our method are as follows:

a) Our proposed method is more general than the various existing methods [20–23] for finding exact compexiton solutions of nonlinear partial differential equations. The demand and achievement of the method lies in the fact that writing the exact solutions of a nonlinear equation as polynomials of φ and ψ with different traveling wave variables, whose derivations are in closed form. The consistency of the method and the cutback in the size of computational domain furnish this method a wider applicability.
b) Using two different Riccati equations with different parameters as well as different traveling wave variables as double sub-equations, we can frankly see that when $p_1 = p_2$ or $q_1 = q_2$, φ and ψ satisfy different Riccati equations, so hyperbolic functions and triangular functions can become visible in one solution at the same time with different traveling wave variables. These types of complexiton solutions have not been gained by any other Riccati equation method [20–23].
c) Ma [20] investigated the complexiton solutions to the KdV equation throughout its bilinear form. On the other hand, using a mapping relation, Lou et al. [23] originated some new types of complexiton solutions. Even if our method cannot recuperate all of the

complexiton solutions achieved by Ma's method and Lou's method, supplementary new types of complexiton solutions cannot be originated by their methods. In fastidious, our method is a unified undemanding and easy algebraic algorithm for both integrable and non-integrable equations.

Application of Our Method

To establish validity and effectiveness of our method, we handle this method in the two nonlinear evolution equations namely the (1+1)-dimensional Burger equation and the (1+1)-dimensional Gardner equation (or combined KdV–mKdV).

Example-1

Let us consider the (1 + 1)-dimensional Burger equation [24,25], in the form

$$u_t + 2uu_x - u_{xx} = 0, \tag{11}$$

Burgers equation (11) is a model for nonlinear wave propagation, especially in fluid mechanics. The equation arises in various characteristic areas of applied mathematics, such as modeling of gas dynamics and traffic flow.

According to the method, we assume that the solutions of Eq. (11) are as follows:

$$u(x,t) = b_0 + \frac{b_1\varphi(\xi) + b_2\psi(\eta)}{b_3 + b_4\varphi(\xi)\psi(\eta)}, \tag{12}$$

where $b_0, b_1, b_2, b_3, b_4, \xi = k_1x + w_1t$ and $\eta = k_2x + w_2t$ are arbitrary nonzero constants.

Substituting Eq. (12) into Eq. (11) along with Eqs. (3) and (4) yields a system of equations with respect to $\varphi^m\psi^n$, $(m = 0, 1, 2, \ldots, n = 0, 1, 2, \ldots)$, then set all coefficients of $\varphi^m\psi^n$ in the obtained system of equations to be zero, we obtain a set of over-determined PDEs with respect to $b_0, b_1, b_2, b_3, b_4, k_1, w_1, k_2, w_2$.

Solving the over-determined PDEs by use of Maple, we can obtain the following results.

Case 1.

$$\begin{cases} b_0 = b_0, b_1 = 0, b_2 = -b_4k_1q_1, b_3 = 0, b_4 = b_4, \\ w_1 = -2b_0k_1, w_2 = w_2. \end{cases} \tag{13}$$

Case 2.

$$\begin{cases} b_0 = b_0, b_1 = -b_4k_2q_2, b_2 = 0, b_3 = 0, b_4 = b_4, \\ w_1 = w_1, w_2 = -2b_0k_2. \end{cases} \tag{14}$$

Case 3.

$$\begin{cases} b_0 = b_0, b_1 = 0, b_2 = b_3k_2p_2, b_3 = b_3, b_4 = 0, \\ w_1 = w_1, w_2 = -2b_0k_2. \end{cases} \tag{15}$$

Case 4.

$$\begin{cases} b_0 = b_0, b_1 = b_1, b_2 = 0, b_3 = \frac{q_1}{k_1p_1}, b_4 = 0, \\ w_1 = -2b_0k_1, w_2 = w_2. \end{cases} \tag{16}$$

Case 5.

$$\begin{cases} b_0 = b_0, b_1 = b_2\sqrt{\frac{p_1q_2}{p_2q_1}}, b_2 = b_2, b_3 = \dfrac{b_2\left(k_2q_2+k_1q_1\sqrt{\frac{p_1q_2}{p_2q_1}}\right)}{p_1q_1k_1^2+p_2q_2k_2^2+2k_1q_1k_2p_2\sqrt{\frac{p_1q_2}{p_2q_1}}}, \\ b_4 = \dfrac{p_1b_2\left(k_2q_2+k_1q_1\sqrt{\frac{p_1q_2}{p_2q_1}}\right)}{q_1\left(p_1q_1k_1^2\sqrt{\frac{p_1q_2}{p_2q_1}}+p_2q_2k_2^2\sqrt{\frac{p_1q_2}{p_2q_1}}+2k_1p_1k_2q_2\right)}, w_1 = -2b_0k_1, w_2 = -2b_0k_2. \end{cases} \tag{17}$$

Case 6.

$$\begin{cases} b_0 = b_0, b_1 = b_2\frac{\sqrt{p_1q_1p_2q_2}}{p_2q_1}, b_2 = b_2, b_3 = \frac{b_2\left(\frac{\sqrt{p_1q_1p_2q_2}}{p_2q_1}k_1q_1+k_2q_2\right)}{p_1q_1k_1^2+p_2q_2k_2^2+2k_1q_1k_2p_2\frac{\sqrt{p_1q_1p_2q_2}}{p_2q_1}}, \\ b_4 = -\frac{p_1b_2\left(\frac{\sqrt{p_1q_1p_2q_2}}{p_2q_1}k_1q_1+k_2q_2\right)}{q_1\left(p_1q_1k_1^2\frac{\sqrt{p_1q_1p_2q_2}}{p_2q_1}+p_2q_2k_2^2\frac{\sqrt{p_1q_1p_2q_2}}{p_2q_1}+2k_1p_1k_2q_2\right)}, w_1 = w_1, \\ w_2 = -\frac{w_1p_1+2b_0\left(\frac{\sqrt{p_1q_1p_2q_2}}{p_2q_1}\right)k_2p_2+2b_0k_1p_1}{p_2\frac{\sqrt{p_1q_1p_2q_2}}{p_2q_1}}. \end{cases} \tag{18}$$

Note that: Since the solutions obtained here are so many with complexitons and without complexitons, we just write some new and complexiton solutions for the Burgers equation to demonstrate the effectiveness of our method.

Using (17), one can get various types of complexiton solutions of Eq. (1) as follows:
Family-1: When $b_0 = b_2 = const.$, $q_1 = 1$, $p_1 = -1$, then we can get some complexiton solutions:

i. When $q_2 = p_2 = \frac{1}{2}$, then

$$u_{1,2} = a_0 \mp \left(\frac{1}{2}k_2 \pm Ik_1\right)\left\{\frac{I\tanh(\xi) \mp (\sec(\eta) - \tan(\eta)}{1 \pm I\tanh(\xi)(\sec(\eta) - \tan(\eta))}\right\}$$

$$u_{3,4} = a_0 \mp \left(\frac{1}{2}k_2 \pm Ik_1\right)\left\{\frac{I\tanh(\xi) \pm (\csc(\eta) - \cot(\eta)}{1 \mp I\tanh(\xi)(\csc(\eta) - \cot(\eta)}\right\}$$

$$u_{5,6} = a_0 \mp \left(\frac{1}{2}k_2 \pm Ik_1\right)\left\{\frac{I\coth(\xi) \mp (\sec(\eta) - \tan(\eta)}{1 \pm I\coth(\xi)(\sec(\eta) - \tan(\eta)}\right\}$$

$$u_{7,8} = a_0 \mp \left(\frac{1}{2}k_2 \pm Ik_1\right)\left\{\frac{I\coth(\xi) \pm (\csc(\eta) - \cot(\eta)}{1 \mp I\coth(\xi)(\csc(\eta) - \cot(\eta)}\right\}$$

ii. When $q_2 = p_2 = -\frac{1}{2}$, then

$$u_{9,10} = a_0 \pm \left(\frac{1}{2}k_2 \mp Ik_1\right)\left\{\frac{I\tanh(\xi) \mp (\sec(\eta) + \tan(\eta)}{1 \pm I\tanh(\xi)(\sec(\eta) + \tan(\eta)}\right\}$$

$$u_{11,12} = a_0 \pm \left(\frac{1}{2}k_2 \mp Ik_1\right)\left\{\frac{I\tanh(\xi) \pm (\csc(\eta) + \cot(\eta)}{1 \mp I\tanh(\xi)(\csc(\eta) + \cot(\eta)}\right\}$$

$$u_{13,14} = a_0 \pm \left(\frac{1}{2}k_2 \mp Ik_1\right)\left\{\frac{I\coth(\xi) \mp (\sec(\eta) + \tan(\eta)}{1 \pm I\coth(\xi)(\sec(\eta) + \tan(\eta)}\right\}$$

$$u_{15,16} = a_0 \pm \left(\frac{1}{2}k_2 \mp Ik_1\right)\left\{\frac{I\coth(\xi) \pm (\csc(\eta) + \cot(\eta)}{1 \mp I\coth(\xi)(\csc(\eta) + \cot(\eta)}\right\}$$

iii. When $q_2 = p_2 = 1$, then

$$u_{17,18} = a_0 \mp (k_2 \pm Ik_1)\left\{\frac{I\tanh(\xi) \pm \tan(\eta)}{1 \mp I\tanh(\xi)\tan(\eta)}\right\}$$

$$u_{19,20} = a_0 \mp (k_2 \pm Ik_1)\left\{\frac{I\coth(\xi) \pm \tan(\eta)}{1 \mp I\coth(\xi)\tan(\eta)}\right\}$$

iv. When $q_2 = p_2 = -1$, then

$$u_{21,22} = a_0 \pm (k_2 \mp Ik_1)\left\{\frac{I\tanh(\xi) \pm \cot(\eta)}{1 \mp I\tanh(\xi)\cot(\eta)}\right\}$$

$$u_{23,24} = a_0 \pm (k_2 \mp I k_1) \left\{ \frac{I \coth(\xi) \pm \cot(\eta)}{1 \mp I \coth(\xi) \cot(\eta)} \right\}$$

v. When $q_2 = \frac{1}{2}$, $p_2 = -\frac{1}{2}$, then

$$u_{25,26} = a_0 \pm \left(\frac{1}{2}k_2 \pm k_1\right) \left\{ \frac{\tanh(\xi) \pm \left(\tanh(\eta) - \frac{I}{\cosh(\eta)}\right)}{1 \pm \tanh(\xi)\left(\tanh(\eta) - \frac{I}{\cosh(\eta)}\right)} \right\}$$

$$u_{27,28} = a_0 \pm \left(\frac{1}{2}k_2 \pm k_1\right) \left\{ \frac{\tanh(\xi) \pm \left(\tanh(\eta) + \frac{I}{\cosh(\eta)}\right)}{1 \pm \tanh(\xi)\left(\tanh(\eta) + \frac{I}{\cosh(\eta)}\right)} \right\}$$

$$u_{29,30} = a_0 \pm \left(\frac{1}{2}k_2 \pm k_1\right) \left\{ \frac{\tanh(\xi) \pm \left(\coth(\eta) + \frac{1}{\sinh(\eta)}\right)}{1 \pm \tanh(\xi)\left(\coth(\eta) + \frac{1}{\sinh(\eta)}\right)} \right\}$$

$$u_{31,32} = a_0 \pm \left(\frac{1}{2}k_2 \pm k_1\right) \left\{ \frac{\tanh(\xi) \pm \left(\coth(\eta) - \frac{1}{\sinh(\eta)}\right)}{1 \pm \tanh(\xi)\left(\coth(\eta) - \frac{1}{\sinh(\eta)}\right)} \right\}$$

$$u_{33,34} = a_0 \pm \left(\frac{1}{2}k_2 \pm k_1\right) \left\{ \frac{\coth(\xi) \pm \left(\tanh(\eta) - \frac{I}{\cosh(\eta)}\right)}{1 \pm \coth(\xi)\left(\tanh(\eta) - \frac{I}{\cosh(\eta)}\right)} \right\}$$

$$u_{35,36} = a_0 \pm \left(\frac{1}{2}k_2 \pm k_1\right) \left\{ \frac{\coth(\xi) \pm \left(\tanh(\eta) + \frac{I}{\cosh(\eta)}\right)}{1 \pm \coth(\xi)\left(\tanh(\eta) + \frac{I}{\cosh(\eta)}\right)} \right\}$$

$$u_{37,38} = a_0 \pm \left(\frac{1}{2}k_2 \pm k_1\right) \left\{ \frac{\coth(\xi) \pm \left(\coth(\eta) + \frac{1}{\sinh(\eta)}\right)}{1 \pm \coth(\xi)\left(\coth(\eta) + \frac{1}{\sinh(\eta)}\right)} \right\}$$

$$u_{39,40} = a_0 \pm \left(\frac{1}{2}k_2 \pm k_1\right) \left\{ \frac{\coth(\xi) \pm \left(\coth(\eta) - \frac{1}{\sinh(\eta)}\right)}{1 \pm \coth(\xi)\left(\coth(\eta) - \frac{1}{\sinh(\eta)}\right)} \right\}$$

where $\xi = k_1 x + w_1 t$ and $\eta = k_2 x + w_2 t$, $w_1 = -2b_0 k_1$, $w_2 = -2b_0 k_2$.
Family-2: When $b_0 = b_2 = const.$, $q_1 = p_1 = \pm\frac{1}{2}$, then we can get some complexiton solutions:

i. When $q_2 = p_2 = 1$, then

$$u_{41,42} = a_0 \mp \left(\frac{1}{2}k_2 \pm k_1\right) \left\{ \frac{\tan(\xi) \mp (\sec(\eta) - \tan(\eta))}{1 \pm \tan(\xi)(\sec(\eta) - \tan(\eta))} \right\}$$

$$u_{43,44} = a_0 \mp \left(\frac{1}{2}k_2 \pm k_1\right) \left\{ \frac{\tan(\xi) \pm (\csc(\eta) - \cot(\eta))}{1 \mp \tan(\xi)(\csc(\eta) - \cot(\eta))} \right\}$$

$$u_{45,46} = a_0 \pm \left(\frac{1}{2}k_2 \mp k_1\right) \left\{ \frac{\tan(\xi) \mp (\sec(\eta) + \tan(\eta))}{1 \pm \tan(\xi)(\sec(\eta) + \tan(\eta))} \right\}$$

$$u_{47,48} = a_0 \pm \left(\frac{1}{2}k_2 \mp k_1\right) \left\{ \frac{\tan(\xi) \pm (\csc(\eta) + \cot(\eta))}{1 \mp \tan(\xi)(\csc(\eta) + \cot(\eta))} \right\}$$

where $\xi = k_1 x + w_1 t$ and $\eta = k_2 x + w_2 t$, $w_1 = -2b_0 k_1$, $w_2 = -2b_0 k_2$.

Family-3: When $b_0 = b_2 = const., q_1 = p_1 = 1$, then we can get some complexiton solutions:

i When $q_2 = p_2 = -1$, then

$$u_{49,50} = a_0 \pm (k_2 \mp k_1) \left\{ \frac{\tan(\xi) \pm \cot(\eta)}{1 \mp \tan(\xi)\cot(\eta)} \right\}$$

ii. When $q_2 = \frac{1}{2}$, $p_2 = -\frac{1}{2}$, then

$$u_{51,52} = a_0 \pm \left(\frac{1}{2}k_2 \pm Ik_1\right) \left\{ \frac{I\tan(\xi) \pm \left(\tanh(\eta) - \frac{I}{\cosh(\eta)}\right)}{1 \pm I\tan(\xi)\left(\tanh(\eta) - \frac{I}{\cosh(\eta)}\right)} \right\}$$

$$u_{53,54} = a_0 \pm \left(\frac{1}{2}k_2 \pm Ik_1\right) \left\{ \frac{I\tan(\xi) \pm \left(\tanh(\eta) + \frac{I}{\cosh(\eta)}\right)}{1 \pm I\tan(\xi)\left(\tanh(\eta) + \frac{I}{\cosh(\eta)}\right)} \right\}$$

$$u_{55,56} = a_0 \pm \left(\frac{1}{2}k_2 \pm Ik_1\right) \left\{ \frac{I\tan(\xi) \pm \left(\coth(\eta) + \frac{1}{\sinh(\eta)}\right)}{1 \pm I\tan(\xi)\left(\coth(\eta) + \frac{1}{\sinh(\eta)}\right)} \right\}$$

$$u_{57,58} = a_0 \pm \left(\frac{1}{2}k_2 \pm Ik_1\right) \left\{ \frac{I\tan(\xi) \pm \left(\coth(\eta) - \frac{1}{\sinh(\eta)}\right)}{1 \pm I\tan(\xi)\left(\coth(\eta) - \frac{1}{\sinh(\eta)}\right)} \right\}$$

where $\xi = k_1x + w_1t$ and $\eta = k_2x + w_2t$, $w_1 = -2b_0k_1$, $w_2 = -2b_0k_2$.

Again, using (18), one can get various types of complexiton solutions of Eq. (1) as follows:

Family-1: When $b_0 = b_2 = w_1 = const., q_1 = 1, p_1 = -1$, then we can get some complexiton solutions:

i. When $q_2 = p_2 = \frac{1}{2}$, then

$$u_{1,2} = a_0 \mp \left(\frac{1}{2}k_2 \pm Ik_1\right) \left\{ \frac{I\tanh(\xi) \mp (\sec(\eta) - \tan(\eta)}{1 \pm I\tanh(\xi)(\sec(\eta) - \tan(\eta))} \right\}$$

$$u_{3,4} = a_0 \mp \left(\frac{1}{2}k_2 \pm Ik_1\right) \left\{ \frac{I\tanh(\xi) \pm (\csc(\eta) - \cot(\eta)}{1 \mp I\tanh(\xi)(\csc(\eta) - \cot(\eta)} \right\}$$

$$u_{5,6} = a_0 \mp \left(\frac{1}{2}k_2 \pm Ik_1\right) \left\{ \frac{I\coth(\xi) \mp (\sec(\eta) - \tan(\eta)}{1 \pm I\coth(\xi)(\sec(\eta) - \tan(\eta)} \right\}$$

$$u_{7,8} = a_0 \mp \left(\frac{1}{2}k_2 \pm Ik_1\right) \left\{ \frac{I\coth(\xi) \pm (\csc(\eta) - \cot(\eta)}{1 \mp I\coth(\xi)(\csc(\eta) - \cot(\eta)} \right\}$$

ii. When $q_2 = p_2 = -\frac{1}{2}$, then

$$u_{9,10} = a_0 \mp \left(\frac{1}{2}k_2 \pm Ik_1\right) \left\{ \frac{I\tanh(\xi) \pm (\sec(\eta) + \tan(\eta)}{1 \mp I\tanh(\xi)(\sec(\eta) + \tan(\eta)} \right\}$$

$$u_{11,12} = a_0 \mp \left(\frac{1}{2}k_2 \pm Ik_1\right) \left\{ \frac{I\tanh(\xi) \mp (\csc(\eta) + \cot(\eta)}{1 \pm I\tanh(\xi)(\csc(\eta) + \cot(\eta)} \right\}$$

$$u_{13,14} = a_0 \mp \left(\frac{1}{2}k_2 \pm Ik_1\right) \left\{ \frac{I\coth(\xi) \pm (\sec(\eta) + \tan(\eta)}{1 \mp I\coth(\xi)(\sec(\eta) + \tan(\eta)} \right\}$$

$$u_{15,16} = a_0 \mp \left(\frac{1}{2}k_2 \pm Ik_1\right) \left\{ \frac{I\coth(\xi) \mp (\csc(\eta) + \cot(\eta)}{1 \pm I\coth(\xi)(\csc(\eta) + \cot(\eta)} \right\}$$

iii. When $q_2 = p_2 = 1$, then

$$u_{17,18} = a_0 \mp (k_2 \pm I k_1) \left\{ \frac{I \tanh(\xi) \pm \tan(\eta)}{1 \mp I \tanh(\xi) \tan(\eta)} \right\}$$
$$u_{19,20} = a_0 \mp (k_2 \pm I k_1) \left\{ \frac{I \coth(\xi) \pm \tan(\eta)}{1 \mp I \coth(\xi) \tan(\eta)} \right\}$$

iv. When $q_2 = p_2 = -1$, then

$$u_{21,22} = a_0 \mp (k_2 \pm I k_1) \left\{ \frac{I \tanh(\xi) \mp \cot(\eta)}{1 \pm I \tanh(\xi) \cot(\eta)} \right\}$$
$$u_{23,24} = a_0 \mp (k_2 \pm I k_1) \left\{ \frac{I \coth(\xi) \mp \cot(\eta)}{1 \pm I \coth(\xi) \cot(\eta)} \right\}$$

v. When $q_2 = \frac{1}{2}$, $p_2 = -\frac{1}{2}$, then

$$u_{25,26} = a_0 \mp \left(\frac{1}{2}k_2 \mp k_1\right) \left\{ \frac{\tanh(\xi) \mp \left(\tanh(\eta) - \frac{I}{\cosh(\eta)}\right)}{1 \mp \tanh(\xi) \left(\tanh(\eta) - \frac{I}{\cosh(\eta)}\right)} \right\}$$

$$u_{27,28} = a_0 \mp \left(\frac{1}{2}k_2 \mp k_1\right) \left\{ \frac{\tanh(\xi) \mp \left(\tanh(\eta) + \frac{I}{\cosh(\eta)}\right)}{1 \mp \tanh(\xi) \left(\tanh(\eta) + \frac{I}{\cosh(\eta)}\right)} \right\}$$

$$u_{29,30} = a_0 \mp \left(\frac{1}{2}k_2 \mp k_1\right) \left\{ \frac{\tanh(\xi) \mp \left(\coth(\eta) + \frac{1}{\sinh(\eta)}\right)}{1 \mp \tanh(\xi) \left(\coth(\eta) + \frac{1}{\sinh(\eta)}\right)} \right\}$$

$$u_{31,32} = a_0 \mp \left(\frac{1}{2}k_2 \mp k_1\right) \left\{ \frac{\tanh(\xi) \mp \left(\coth(\eta) - \frac{1}{\sinh(\eta)}\right)}{1 \mp \tanh(\xi) \left(\coth(\eta) - \frac{1}{\sinh(\eta)}\right)} \right\}$$

$$u_{33,34} = a_0 \mp \left(\frac{1}{2}k_2 \mp k_1\right) \left\{ \frac{\coth(\xi) \mp \left(\tanh(\eta) - \frac{I}{\cosh(\eta)}\right)}{1 \mp \coth(\xi) \left(\tanh(\eta) - \frac{I}{\cosh(\eta)}\right)} \right\}$$

$$u_{35,36} = a_0 \mp \left(\frac{1}{2}k_2 \mp k_1\right) \left\{ \frac{\coth(\xi) \mp \left(\tanh(\eta) + \frac{I}{\cosh(\eta)}\right)}{1 \mp \coth(\xi) \left(\tanh(\eta) + \frac{I}{\cosh(\eta)}\right)} \right\}$$

$$u_{37,38} = a_0 \mp \left(\frac{1}{2}k_2 \mp k_1\right) \left\{ \frac{\coth(\xi) \mp \left(\coth(\eta) + \frac{1}{\sinh(\eta)}\right)}{1 \mp \coth(\xi) \left(\coth(\eta) + \frac{1}{\sinh(\eta)}\right)} \right\}$$

$$u_{39,40} = a_0 \mp \left(\frac{1}{2}k_2 \mp k_1\right) \left\{ \frac{\coth(\xi) \mp \left(\coth(\eta) - \frac{1}{\sinh(\eta)}\right)}{1 \mp \coth(\xi) \left(\coth(\eta) - \frac{1}{\sinh(\eta)}\right)} \right\}$$

where $\xi = k_1 x + w_1 t$ and $\eta = k_2 x + w_2 t$, $w_1 = w_1$, $w_2 = -\frac{w_1 p_1 + 2b_0 \left(\frac{\sqrt{p_1 q_1 p_2 q_2}}{p_2 q_1}\right) k_2 p_2 + 2b_0 k_1 p_1}{p_2 \frac{\sqrt{p_1 q_1 p_2 q_2}}{p_2 q_1}}$.

Family-2: When $b_0 = b_2 = w_1 = const.$, $q_1 = p_1 = \pm\frac{1}{2}$, then we can get some complexiton solutions:

i. when $q_2 = p_2 = 1$, then

$$u_{41,42} = a_0 \mp \left(\frac{1}{2}k_2 \pm k_1\right)\left\{\frac{\tan(\xi) \mp (\sec(\eta) - \tan(\eta))}{1 \pm \tan(\xi)\,(\sec(\eta) - \tan(\eta))}\right\}$$
$$u_{43,44} = a_0 \mp \left(\frac{1}{2}k_2 \pm k_1\right)\left\{\frac{\tan(\xi) \pm (\csc(\eta) - \cot(\eta))}{1 \mp \tan(\xi)\,(\csc(\eta) - \cot(\eta))}\right\}$$
$$u_{45,46} = a_0 \mp \left(\frac{1}{2}k_2 \pm k_1\right)\left\{\frac{\tan(\xi) \pm (\sec(\eta) + \tan(\eta))}{1 \mp \tan(\xi)\,(\sec(\eta) + \tan(\eta))}\right\}$$
$$u_{47,48} = a_0 \mp \left(\frac{1}{2}k_2 \pm k_1\right)\left\{\frac{\tan(\xi) \mp (\csc(\eta) + \cot(\eta))}{1 \pm \tan(\xi)\,(\csc(\eta) + \cot(\eta))}\right\}$$

ii. when $q_2 = p_2 = -1$, then

$$u_{49,50} = a_0 \mp (k_2 \pm \frac{1}{2}k_1)\left\{\frac{(\sec(\xi) + \tan(\xi)) \mp \cot(\eta)}{1 \pm (\sec(\xi) + \tan(\xi))\cot(\eta)}\right\}$$

iii. when $q_2 = \frac{1}{2}$, $p_2 = -\frac{1}{2}$, then

$$u_{51,52} = a_0 \mp \left(\frac{1}{2}k_2 \mp \frac{1}{2}Ik_1\right)\left\{\frac{I(\csc(\xi) - \cot(\xi)) \mp \left(\tanh(\eta) - \frac{I}{\cosh(\eta)}\right)}{1 \mp I(\csc(\xi) - \cot(\xi))\left(\tanh(\eta) - \frac{I}{\cosh(\eta)}\right)}\right\}$$

where $\xi = k_1x + w_1t$ and $\eta = k_2x + w_2t$, $w_1 = w_1$, $w_2 = -\frac{w_1p_1 + 2b_0\left(\frac{\sqrt{p_1q_1p_2q_2}}{p_2q_1}\right)k_2p_2 + 2b_0k_1p_1}{p_2\frac{\sqrt{p_1q_1p_2q_2}}{p_2q_1}}$.

Family-3: When $b_0 = b_2 = w_1 = const.$, $q_1 = p_1 = 1$, then we can get some complexiton solutions:

i. when $q_2 = p_2 = -1$, then

$$u_{53,54} = a_0 \mp (k_2 \pm k_1)\left\{\frac{\tan(\xi) \mp \cot(\eta)}{1 \pm \tan(\xi)\cot(\eta)}\right\}$$

ii. when $q_2 = \frac{1}{2}$, $p_2 = -\frac{1}{2}$, then

$$u_{55,56} = a_0 \mp \left(\frac{1}{2}k_2 \mp Ik_1\right)\left\{\frac{I\tan(\xi) \mp \left(\tanh(\eta) - \frac{I}{\cosh(\eta)}\right)}{1 \mp I\tan(\xi)\left(\tanh(\eta) - \frac{I}{\cosh(\eta)}\right)}\right\}$$

$$u_{57,58} = a_0 \mp \left(\frac{1}{2}k_2 \mp Ik_1\right)\left\{\frac{I\tan(\xi) \mp \left(\tanh(\eta) + \frac{I}{\cosh(\eta)}\right)}{1 \mp I\tan(\xi)\left(\tanh(\eta) + \frac{I}{\cosh(\eta)}\right)}\right\}$$

$$u_{59,60} = a_0 \mp \left(\frac{1}{2}k_2 \mp Ik_1\right)\left\{\frac{I\tan(\xi) \mp \left(\coth(\eta) + \frac{1}{\sinh(\eta)}\right)}{1 \mp I\tan(\xi)\left(\coth(\eta) + \frac{1}{\sinh(\eta)}\right)}\right\}$$

$$u_{61,62} = a_0 \mp \left(\frac{1}{2}k_2 \mp Ik_1\right)\left\{\frac{I\tan(\xi) \mp \left(\coth(\eta) - \frac{1}{\sinh(\eta)}\right)}{1 \mp I\tan(\xi)\left(\coth(\eta) - \frac{1}{\sinh(\eta)}\right)}\right\}$$

where $\xi = k_1x + w_1t$ and $\eta = k_2x + w_2t$, $w_1 = w_1$, $w_2 = -\frac{w_1p_1 + 2b_0\left(\frac{\sqrt{p_1q_1p_2q_2}}{p_2q_1}\right)k_2p_2 + 2b_0k_1p_1}{p_2\frac{\sqrt{p_1q_1p_2q_2}}{p_2q_1}}$.

Family-4: When $b_0 = b_2 = w_1 = const.$, $q_1 = p_1 = -1$, and $q_2 = \frac{1}{2}$, $p_2 = -\frac{1}{2}$, then we can get a complexiton solution:

$$u_{63,64} = a_0 \pm \left(\frac{1}{2}k_2 \mp I k_1\right)\left\{\frac{I\cot(\xi) \pm \left(\coth(\eta) - \frac{1}{\sinh(\eta)}\right)}{1 \pm I\cot(\xi)\left(\coth(\eta) - \frac{1}{\sinh(\eta)}\right)}\right\}$$

where $\xi = k_1x + w_1t$ and $\eta = k_2x + w_2t$, $w_1 = w_1$, $w_2 = -\frac{w_1p_1 + 2b_0\left(\frac{\sqrt{p_1q_1p_2q_2}}{p_2q_1}\right)k_2p_2 + 2b_0k_1p_1}{p_2\frac{\sqrt{p_1q_1p_2q_2}}{p_2q_1}}$.

Similarly, we can write down the other complexiton solution of Eq. (11) which are omitted for convenience.

Example-2

Secondly, we study the (1 + 1)-dimensional Gardner equation (or combined KdV–mKdV) [27,28], in the form

$$u_t + b_1uu_x + b_2u^2u_x + b_3u_{xxx} = 0, \tag{19}$$

where $u = u(x, t)$ and b_1, b_2, b_3 are arbitrary constants.

This is an significant model to realize the propagation of negative ion acoustic plasma waves [28] and can be derived from the structure of plasma motion equations in one dimension with arbitrarily charged cold ions and inertia neglected isothermal electrons. This equation can also be a good explanation of internal waves with large amplitudes [29].

According to the method, we assume that the solutions of Eq. (19) are as follows:

$$u(x, t) = a_0 + \frac{a_1\varphi(\xi) + a_2\psi(\eta)}{a_3 + a_4\varphi(\xi)\psi(\eta)}, \tag{20}$$

where a_0, a_1, a_2, a_3, a_4, $\xi = k_1x + w_1t$ and $\eta = k_2x + w_2t$ are arbitrary nonzero constants.

Substituting Eq. (20) into Eq. (19) along with Eq. (3) and Eq. (4) yields a system of equations with respect to $\varphi^m\psi^n$, ($m = 0, 1, 2, ..., n = 0, 1, 2, ...$), then set all coefficients of $\varphi^m\psi^n$ in the obtained system of equations to be zero, we obtain a set of over-determined PDEs with respect to $a_0, a_1, a_2, a_3, a_4, k_1, w_1, k_2, w_2$.

Solving the over-determined PDEs by use of Maple, we can obtain the following results.

Case 1.

$$\begin{cases} a_0 = -\frac{1}{2}\frac{b_1}{b_2}, a_1 = 0, a_2 = \Delta_1 q_1 k_1 a_4, a_3 = 0, a_4 = a_4, \\ w_1 = \frac{1}{4}\Delta_2, w_2 = w_2. \end{cases} \tag{21}$$

Case 2.

$$\begin{cases} a_0 = -\frac{1}{2}\frac{b_1}{b_2}, a_1 = 0, a_2 = \Delta_1 a_3 k_2 p_2, a_3 = a_3, a_4 = 0, \\ w_1 = w_1, w_2 = \frac{1}{4}\Delta_3. \end{cases} \tag{22}$$

Case 3.

$$\begin{cases} a_0 = -\frac{1}{2}\frac{b_1}{b_2}, a_1 = \Delta_1 k_2 q_2 a_4, a_2 = 0, a_3 = 0, a_4 = a_4, \\ w_1 = w_1, w_2 = \frac{1}{4}\Delta_3. \end{cases} \tag{23}$$

Case 4.

$$\begin{cases} a_0 = -\frac{1}{2}\frac{b_1}{b_2}, a_1 = \Delta_1 a_3 k_1 p_1, a_2 = 0, a_3 = a_3, a_4 = 0, \\ w_1 = \frac{1}{4}\Delta_2, w_2 = w_2. \end{cases} \tag{24}$$

Case 5.

$$\begin{cases} a_0=-\frac{1}{2}\frac{b_1}{b_2},\ a_1=\gamma a_4,\ a_2=\frac{-12\gamma a_4 p_2 b_3 q_2 k_1 q_1 k_2}{(6b_3p_2q_2^2k_2^2+p_2b_2\gamma^2+6q_2q_1k_1^2p_1b_3)},\\ a_3=\frac{12b_3a_4q_2^2k_1q_1k_2}{(6b_3p_2q_2^2k_2^2+p_2b_2\gamma+6q_2q_1k_1^2p_1b_3)},\ a_4=a_4,\\ w_1=\frac{1}{4}\frac{\begin{array}{l}k_1(-8p_2b_2^2k_1^2\gamma^2b_3p_1q_1+p_2b_2\gamma^2b_1^2-24p_2^2b_2^2q_2\gamma^2b_3k_2^2+48b_3q_2^2p_2b_2k_2w_2\\ -6b_3q_2^2p_2b_1^2k_2^2-48b_3^2q_2^2p_2^2b_2k_2^4\\ 96b_3^2q_2^2p_2b_2p_1q_1k_2^2-48b_3^2q_2k_1^4b_2q_1^2p_1^2+6b_3q_2k_1^2b_1^2q_1p_1)\end{array}}{b_2(6b_3p_2q_2^2+p_2b_2\gamma^2+6q_1k_1^2p_1q_2b_3)},\\ w_2=w_2. \end{cases} \tag{25}$$

where $\Delta_1=\frac{\sqrt{-6b_2b_3}}{b_2}$, $\Delta_2=\frac{k_1(-8b_3p_1k_1^2q_1b_2+b_1^2)}{b_2}$, $\Delta_3=\frac{k_2(-8b_3p_2k_2^2q_2b_2+b_1^2)}{b_2}$ and $\gamma=\frac{\sqrt{-6p_2b_2b_3q_2(q_2k_2^2p_2+q_1k_1^2p_1+2k_1k_2\sqrt{p_2p_1q_1q_2})}}{p_2b_2}$

Note that: Since the solutions obtained here are so many with complexitons and without complexitons, we just write complexiton solutions for the (1 + 1)-dimensional Gardner equation (or combined KdV–mKdV) equation.

Using (25), one can get various types of complexiton solutions of Eq. (1) as follows:

Family-1: When $b_0=b_2=w_1=const.$, $q_1=1$, $p_1=-1$, then we can get some complexiton solutions:

i. When $q_2=p_2=\frac{1}{2}$, then

$$u_{1,2}=a_0+\frac{\left\{\pm\frac{1}{b_2}\left(\alpha_1a_4\tanh(k_1x+\frac{1}{4\beta_1}\delta_1t)\mp\frac{3\alpha_1a_4b_3k_1k_2(\sec(\eta)+\tan(\eta))}{b_2\beta_1}\right)\right\}}{\left(\frac{3b_3a_4k_1k_2}{\frac{3}{4}b_3k_2^2-3k_1^2b_3\pm\frac{1}{2}\alpha_1}+a_4\tanh(k_1x+\frac{1}{4\beta_1}\delta_1t)\right)(\sec(\eta)+\tan(\eta))}$$

$$u_{3,4}=a_0+\frac{\left\{\pm\frac{1}{b_2}\left(\alpha_1a_4\tanh(k_1x+\frac{1}{4\beta_1}\delta_1t)\mp\frac{3\alpha_1a_4b_3k_1k_2(\csc(\eta)-\cot(\eta))}{b_2\beta_1}\right)\right\}}{\left(\frac{3b_3a_4k_1k_2}{\frac{3}{4}b_3k_2^2-3k_1^2b_3\pm\frac{1}{2}\alpha_1}+a_4\tanh(k_1x+\frac{1}{4\beta_1}\delta_1t)\right)(\csc(\eta)-\cot(\eta))}$$

$$u_{5,6}=a_0+\frac{\left\{\pm\frac{1}{b_2}\left(\alpha_1a_4\coth(k_1x+\frac{1}{4\beta_1}\delta_1t)\mp\frac{3\alpha_1a_4b_3k_1k_2(\sec(\eta)+\tan(\eta))}{b_2\beta_1}\right)\right\}}{\left(\frac{3b_3a_4k_1k_2}{\frac{3}{4}b_3k_2^2-3k_1^2b_3\pm\frac{1}{2}\alpha_1}+a_4\coth(k_1x+\frac{1}{4\beta_1}\delta_1t)\right)(\sec(\eta)+\tan(\eta))}$$

$$u_{7,8}=a_0+\frac{\left\{\pm\frac{1}{b_2}\left(\alpha_1a_4\coth(k_1x+\frac{1}{4\beta_1}\delta_1t)\mp\frac{3\alpha_1a_4b_3k_1k_2(\csc(\eta)-\cot(\eta))}{b_2\beta_1}\right)\right\}}{\left(\frac{3b_3a_4k_1k_2}{\frac{3}{4}b_3k_2^2-3k_1^2b_3\pm\frac{1}{2}\alpha_1}+a_4\coth(k_1x+\frac{1}{4\beta_1}\delta_1t)\right)(\csc(\eta)-\cot(\eta))}$$

where, $a_o=-\frac{1}{2}\frac{b_1}{b_2}$, $\alpha_1=\sqrt{-6b_2b_3\left(\frac{1}{2}k_2+Ik_1\right)^2}$, $\beta_1=\frac{3}{4}b_3k_2^2-3k_1^2b_3-3b_3\left(\frac{1}{2}k_2+Ik_1\right)^2$ and $\delta_1=k_1(-24b_2k_1^2b_3^2\left(\frac{1}{2}k_2+Ik_1\right)^2-3b_3\left(\frac{1}{2}k_2+Ik_1\right)^2b_1^2+18b_2b_3^2\left(\frac{1}{2}k_2+Ik_1\right)^2k_2^2+6b_3b_2k_2w_2-\frac{3}{4}b_3b_1^2k_2^2-\frac{3}{2}b_3^2b_2k_2^4-12b_3^2k_1^2b_2k_2^2-24b_3^2k_1^4b_2-3b_3k_1^2b_1^2)$

ii. When $q_2=p_2=-\frac{1}{2}$, then

$$u_{9,10}=a_0+\frac{\left\{\mp\frac{1}{b_2}\left(\alpha_2a_4\tanh(k_1x+\frac{1}{4\beta_2}\delta_2t)\pm\frac{3\alpha_2a_4b_3k_1k_2(\sec(\eta)-\tan(\eta))}{b_2\beta_2}\right)\right\}}{\left(\frac{3b_3a_4k_1k_2}{-\frac{3}{4}b_3k_2^2+3k_1^2b_3\pm\frac{1}{2}\alpha_2}+a_4\tanh(k_1x+\frac{1}{4\beta_1}\delta_1t)\right)(\sec(\eta)-\tan(\eta))}$$

$$u_{11,12} = a_0 + \frac{\left\{\mp\frac{1}{b_2}\left(\alpha_2 a_4 \tanh(k_1 x + \frac{1}{4\beta_2}\delta_2 t) \pm \frac{3\alpha_2 a_4 b_3 k_1 k_2(\csc(\eta)+\cot(\eta))}{b_2\beta_2}\right)\right\}}{\left(\frac{3b_3 a_4 k_1 k_2}{-\frac{3}{4}b_3 k_2^2 + 3k_1^2 b_3 \pm \frac{1}{2}\alpha_2} + a_4 \tanh(k_1 x + \frac{1}{4\beta_1}\delta_1 t)\right)(\csc(\eta)+\cot(\eta))}$$

$$u_{13,14} = a_0 + \frac{\left\{\mp\frac{1}{b_2}\left(\alpha_2 a_4 \coth(k_1 x + \frac{1}{4\beta_2}\delta_2 t) \pm \frac{3\alpha_2 a_4 b_3 k_1 k_2(\sec(\eta)-\tan(\eta))}{b_2\beta_2}\right)\right\}}{\left(\frac{3b_3 a_4 k_1 k_2}{-\frac{3}{4}b_3 k_2^2 + 3k_1^2 b_3 \pm \frac{1}{2}\alpha_2} + a_4 \coth(k_1 x + \frac{1}{4\beta_1}\delta_1 t)\right)(\sec(\eta)-\tan(\eta))}$$

$$u_{15,16} = a_0 + \frac{\left\{\mp\frac{1}{b_2}\left(\alpha_2 a_4 \coth(k_1 x + \frac{1}{4\beta_2}\delta_2 t) \pm \frac{3\alpha_2 a_4 b_3 k_1 k_2(\csc(\eta)+\cot(\eta))}{b_2\beta_2}\right)\right\}}{\left(\frac{3b_3 a_4 k_1 k_2}{-\frac{3}{4}b_3 k_2^2 + 3k_1^2 b_3 \pm \frac{1}{2}\alpha_2} + a_4 \coth(k_1 x + \frac{1}{4\beta_1}\delta_1 t)\right)(\csc(\eta)+\cot(\eta))}$$

where, $a_o = -\frac{1}{2}\frac{b_1}{b_2}, \alpha_2 = \sqrt{-6b_2 b_3\left(\frac{1}{2}k_2 + Ik_1\right)^2}, \beta_2 = -\frac{3}{4}b_3 k_2^2 + 3k_1^2 b_3 + 3b_3\left(\frac{1}{2}k_2 + Ik_1\right)^2$ and $\delta_2 = k_1(24b_2 k_1^2 b_3^2\left(\frac{1}{2}k_2 + Ik_1\right)^2 + 3b_3\left(\frac{1}{2}k_2 + Ik_1\right)^2 b_1^2 - 18b_2 b_3^2\left(\frac{1}{2}k_2 + Ik_1\right)^2 k_2^2 - 6b_3 b_2 k_2 w_2 + \frac{3}{4}b_3 b_1^2 k_2^2 + \frac{3}{2}b_3^2 b_2 k_2^4 + 12b_3^2 k_1^2 b_2 k_2^2 + 24b_3^2 k_1^4 b_2 + 3b_3 k_1^2 b_1^2)$

iii. When $q_2 = p_2 = 1$, then

$$u_{17,18} = a_0 + \frac{\left\{\pm\frac{1}{b_2}\left(\alpha_3 a_4 \tanh(k_1 x + \frac{1}{4\beta_3}\delta_3 t) \mp \frac{12\alpha_3 a_4 b_3 k_1 k_2 \tan(\eta)}{b_2\beta_1}\right)\right\}}{\left(\frac{12b_3 a_4 k_1 k_2}{6b_3 k_2^2 - 6k_1^2 b_3 \pm \alpha_3} + a_4 \tanh(k_1 x + \frac{1}{4\beta_3}\delta_3 t)\right)\tan(\eta)}$$

$$u_{19,20} = a_0 + \frac{\left\{\pm\frac{1}{b_2}\left(\alpha_3 a_4 \coth(k_1 x + \frac{1}{4\beta_3}\delta_3 t) \mp \frac{12\alpha_3 a_4 b_3 k_1 k_2 \tan(\eta)}{b_2\beta_1}\right)\right\}}{\left(\frac{12b_3 a_4 k_1 k_2}{6b_3 k_2^2 - 6k_1^2 b_3 \pm \alpha_3} + a_4 \coth(k_1 x + \frac{1}{4\beta_3}\delta_3 t)\right)\tan(\eta)}$$

where, $a_o = -\frac{1}{2}\frac{b_1}{b_2}, \alpha_3 = \sqrt{-6b_2 b_3 (k_2 + Ik_1)^2}, \beta_3 = 6b_3 k_2^2 - 6k_1^2 b_3 - 6b_3 (k_2 + Ik_1)^2$ and $\delta_3 = k_1(-48b_2 k_1^2 b_3^2 (k_2 + Ik_1)^2 - 6b_3 (k_2 + Ik_1)^2 b_1^2 + 144b_2 b_3^2 (k_2 + Ik_1)^2 k_2^2 + 48b_3 b_2 k_2 w_2 - 6b_3 b_1^2 k_2^2 - 48b_3^2 b_2 k_2^4 - 96b_3^2 k_1^2 b_2 k_2^2 - 48b_3^2 k_1^4 b_2 - 6b_3 k_1^2 b_1^2)$

iv. When $q_2 = p_2 = -1$, then

$$u_{21,22} = a_0 + \frac{\left\{\mp\frac{1}{b_2}\left(\alpha_4 a_4 \tanh(k_1 x + \frac{1}{4\beta_4}\delta_4 t) \pm \frac{12\alpha_3 a_4 b_3 k_1 k_2 \cot(\eta)}{b_2\beta_1}\right)\right\}}{\left(\frac{12b_3 a_4 k_1 k_2}{-6b_3 k_2^2 + 6k_1^2 b_3 \pm \alpha_4} + a_4 \tanh(k_1 x + \frac{1}{4\beta_4}\delta_4 t)\right)\cot(\eta)}$$

$$u_{23,24} = a_0 + \frac{\left\{\mp\frac{1}{b_2}\left(\alpha_4 a_4 \coth(k_1 x + \frac{1}{4\beta_4}\delta_4 t) \pm \frac{12\alpha_3 a_4 b_3 k_1 k_2 \cot(\eta)}{b_2\beta_1}\right)\right\}}{\left(\frac{12b_3 a_4 k_1 k_2}{-6b_3 k_2^2 + 6k_1^2 b_3 \pm \alpha_4} + a_4 \coth(k_1 x + \frac{1}{4\beta_4}\delta_4 t)\right)\cot(\eta)}$$

where, $a_o = -\frac{1}{2}\frac{b_1}{b_2}, \alpha_4 = \sqrt{-6b_2 b_3 (k_2 + Ik_1)^2}, \beta_4 = -6b_3 k_2^2 + 6k_1^2 b_3 + 6b_3 (k_2 + Ik_1)^2$ and $\delta_4 = k_1(48b_2 k_1^2 b_3^2 (k_2 + Ik_1)^2 + 6b_3 (k_2 + Ik_1)^2 b_1^2 - 144b_2 b_3^2 (k_2 + Ik_1)^2 k_2^2 - 48b_3 b_2 k_2 w_2 + 6b_3 b_1^2 k_2^2 + 48b_3^2 b_2 k_2^4 + 96b_3^2 k_1^2 b_2 k_2^2 + 48b_3^2 k_1^4 b_2 + 6b_3 k_1^2 b_1^2)$

v. When $q_2 = \frac{1}{2}$, $p_2 = -\frac{1}{2}$, then

$$u_{25,26} = a_0 + \frac{\left\{\mp\frac{1}{b_2}\left(\alpha_5 a_4 \tanh(k_1 x + \frac{1}{4\beta_5}\delta_5 t) \mp \frac{3\alpha_5 a_4 b_3 k_1 k_2\left(\tanh(\eta)+\frac{I}{\cosh(\eta)}\right)}{b_2\beta_5}\right)\right\}}{\left(\frac{3b_3 a_4 k_1 k_2}{-\frac{3}{4}b_3 k_2^2 - 3k_1^2 b_3 \pm \frac{1}{2}\alpha_5} + a_4 \tanh(k_1 x + \frac{1}{4\beta_5}\delta_5 t)\right)\left(\tanh(\eta) + \frac{I}{\cosh(\eta)}\right)}$$

$$u_{27,28} = a_0 + \frac{\left\{\mp\frac{1}{b_2}\left(\alpha_5 a_4 \tanh(k_1 x + \frac{1}{4\beta_5}\delta_5 t) \mp \frac{3\alpha_5 a_4 b_3 k_1 k_2\left(\tanh(\eta)-\frac{I}{\cosh(\eta)}\right)}{b_2\beta_5}\right)\right\}}{\left(\frac{3b_3 a_4 k_1 k_2}{-\frac{3}{4}b_3 k_2^2 - 3k_1^2 b_3 \pm \frac{1}{2}\alpha_5} + a_4 \tanh(k_1 x + \frac{1}{4\beta_5}\delta_5 t)\right)\left(\tanh(\eta) - \frac{I}{\cosh(\eta)}\right)}$$

$$u_{29,30} = a_0 + \frac{\left\{\mp\frac{1}{b_2}\left(\alpha_5 a_4 \tanh(k_1 x + \frac{1}{4\beta_5}\delta_5 t) \mp \frac{3\alpha_5 a_4 b_3 k_1 k_2\left(\coth(\eta)+\frac{1}{\sinh(\eta)}\right)}{b_2\beta_5}\right)\right\}}{\left(\frac{3b_3 a_4 k_1 k_2}{-\frac{3}{4}b_3 k_2^2 - 3k_1^2 b_3 \pm \frac{1}{2}\alpha_5} + a_4 \tanh(k_1 x + \frac{1}{4\beta_5}\delta_5 t)\right)\left(\coth(\eta) + \frac{1}{\sinh(\eta)}\right)}$$

$$u_{31,32} = a_0 + \frac{\left\{\mp\frac{1}{b_2}\left(\alpha_5 a_4 \tanh(k_1 x + \frac{1}{4\beta_5}\delta_5 t) \mp \frac{3\alpha_5 a_4 b_3 k_1 k_2\left(\coth(\eta)-\frac{1}{\sinh(\eta)}\right)}{b_2\beta_5}\right)\right\}}{\left(\frac{3b_3 a_4 k_1 k_2}{-\frac{3}{4}b_3 k_2^2 - 3k_1^2 b_3 \pm \frac{1}{2}\alpha_5} + a_4 \tanh(k_1 x + \frac{1}{4\beta_5}\delta_5 t)\right)\left(\coth(\eta) - \frac{1}{\sinh(\eta)}\right)}$$

$$u_{33,34} = a_0 + \frac{\left\{\mp\frac{1}{b_2}\left(\alpha_5 a_4 \coth(k_1 x + \frac{1}{4\beta_5}\delta_5 t) \mp \frac{3\alpha_5 a_4 b_3 k_1 k_2\left(\tanh(\eta)+\frac{I}{\cosh(\eta)}\right)}{b_2\beta_5}\right)\right\}}{\left(\frac{3b_3 a_4 k_1 k_2}{-\frac{3}{4}b_3 k_2^2 - 3k_1^2 b_3 \pm \frac{1}{2}\alpha_5} + a_4 \coth(k_1 x + \frac{1}{4\beta_5}\delta_5 t)\right)\left(\tanh(\eta) + \frac{I}{\cosh(\eta)}\right)}$$

$$u_{35,36} = a_0 + \frac{\left\{\mp\frac{1}{b_2}\left(\alpha_5 a_4 \coth(k_1 x + \frac{1}{4\beta_5}\delta_5 t) \mp \frac{3\alpha_5 a_4 b_3 k_1 k_2\left(\tanh(\eta)-\frac{I}{\cosh(\eta)}\right)}{b_2\beta_5}\right)\right\}}{\left(\frac{3b_3 a_4 k_1 k_2}{-\frac{3}{4}b_3 k_2^2 - 3k_1^2 b_3 \pm \frac{1}{2}\alpha_5} + a_4 \coth(k_1 x + \frac{1}{4\beta_5}\delta_5 t)\right)\left(\tanh(\eta) - \frac{I}{\cosh(\eta)}\right)}$$

$$u_{37,38} = a_0 + \frac{\left\{\mp\frac{1}{b_2}\left(\alpha_5 a_4 \coth(k_1 x + \frac{1}{4\beta_5}\delta_5 t) \mp \frac{3\alpha_5 a_4 b_3 k_1 k_2\left(\coth(\eta)+\frac{1}{\sinh(\eta)}\right)}{b_2\beta_5}\right)\right\}}{\left(\frac{3b_3 a_4 k_1 k_2}{-\frac{3}{4}b_3 k_2^2 - 3k_1^2 b_3 \pm \frac{1}{2}\alpha_5} + a_4 \coth(k_1 x + \frac{1}{4\beta_5}\delta_5 t)\right)\left(\coth(\eta) + \frac{1}{\sinh(\eta)}\right)}$$

$$u_{39,40} = a_0 + \frac{\left\{\mp\frac{1}{b_2}\left(\alpha_5 a_4 \coth(k_1 x + \frac{1}{4\beta_5}\delta_5 t) \mp \frac{3\alpha_5 a_4 b_3 k_1 k_2\left(\coth(\eta)-\frac{1}{\sinh(\eta)}\right)}{b_2\beta_5}\right)\right\}}{\left(\frac{3b_3 a_4 k_1 k_2}{-\frac{3}{4}b_3 k_2^2 - 3k_1^2 b_3 \pm \frac{1}{2}\alpha_5} + a_4 \coth(k_1 x + \frac{1}{4\beta_5}\delta_5 t)\right)\left(\coth(\eta) - \frac{1}{\sinh(\eta)}\right)}$$

where, $a_o = -\frac{1}{2}\frac{b_1}{b_2}$, $\alpha_5 = \sqrt{-6b_2 b_3\left(\frac{1}{2}k_2 - k_1\right)^2}$, $\beta_5 = -\frac{3}{4}b_3 k_2^2 - 3k_1^2 b_3 + 3b_3\left(\frac{1}{2}k_2 - k_1\right)^2$ and

$$\delta_5 = k_1(24b_2 k_1^2 b_3^2\left(\frac{1}{2}k_2 - k_1\right)^2 + 3b_3\left(\frac{1}{2}k_2 - k_1\right)^2 b_1^2 + 18b_2 b_3^2\left(\frac{1}{2}k_2 - k_1\right)^2 k_2^2$$

$$-6b_3b_2k_2w_2+\frac{3}{4}b_3b_1^2k_2^2-\frac{3}{2}b_3^2b_2k_2^4+12b_3^2k_1^2b_2k_2^2-24b_3^2k_1^4b_2-3b_3k_1^2b_1^2)$$

with $\eta = k_2x + w_2t$ and

$$w_1=\frac{1}{4}\left(\frac{\begin{array}{l}k_1(-8p_2b_2^2k_1^2\gamma^2b_3p_1q_1+p_2b_2\gamma^2b_1^2-24p_2^2b_2^2q_2\gamma^2b_3k_2^2+48b_3q_2^2p_2b_2k_2w_2\\-6b_3q_2^2p_2b_1^2k_2^2-48b_3^2q_2^2p_2^2b_2k_2^4+\\96b_3^2q_2^2p_2b_2p_1q_1k_2^2-48b_3^2q_2k_1^4b_2q_1^2p_1^2+6b_3q_2k_1^2b_1^2q_1p_1)\end{array}}{b_2(6b_3p_2q_2^2+p_2b_2\gamma^2+6q_1k_1^2p_1q_2b_3)}\right).$$

Graphical Representation

The graphical demonstrations of some of the obtained complexiton solutions for particular values of the arbitrary constants are revealed in Figs. 1, 2, 3, 4, 5, 6, 7 and 8 in the following figures with the aid of commercial software Maple:

Graphs of the Solutions to the Burger,s Equation

The complexiton solutions to the Burger's equations consist with two traveling variables ξ and η expressed in-terms of $\tanh\xi$ and $\tan\eta$, $\sec\eta$; $\tanh\xi$ and $\cot\eta$, cosecη and $\coth\xi$ and $\tan\eta$, $\sec\eta$ gives the kinky -periodic wave. When coefficients of ξ is greater than that of the η gives solution with kinky dominate on periodicity (see Fig. 1) but when coefficients of ξ is smaller than that of the η gives solution with periodicity increases and dominate on kink type (see Fig. 2).

On the other hand, the complexiton solutions consist with two traveling variables ξ and η expressed in-terms of $\coth\xi$ and $\cot\eta$, cosecη; $\coth\xi$ and $\tan\eta$ gives multi-soliton solutions like Fig. 3 of $u_{7,8}$.

The complexiton solutions consist with two traveling variables ξ and η expressed in-terms of $\cot\xi$ and $\cot\eta$, cosecη; $\cot\xi$ and $\tan\eta$ gives double-periodic solutions like Fig. 4 of $u_{41,42}$.

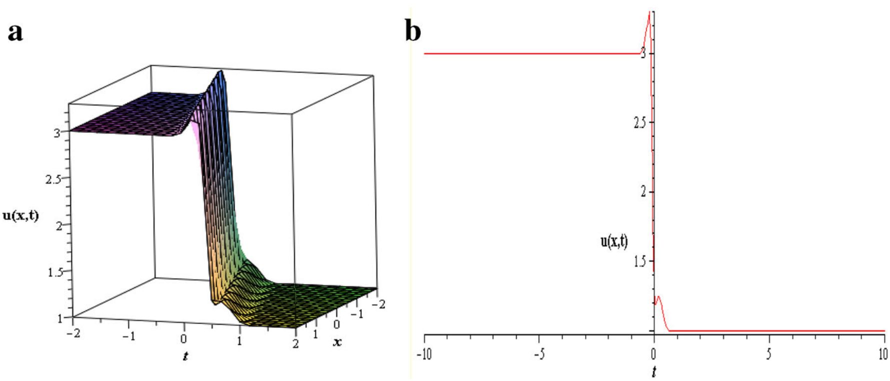

Fig. 1 **a** Kinky-periodic wave solution for $b_0 = 2, k_1 = 1, k_2 = -1.5$ of the real part of $u_{1,2}$ and **b** 2D plot shows the wave propagation pattern at $x = 0$

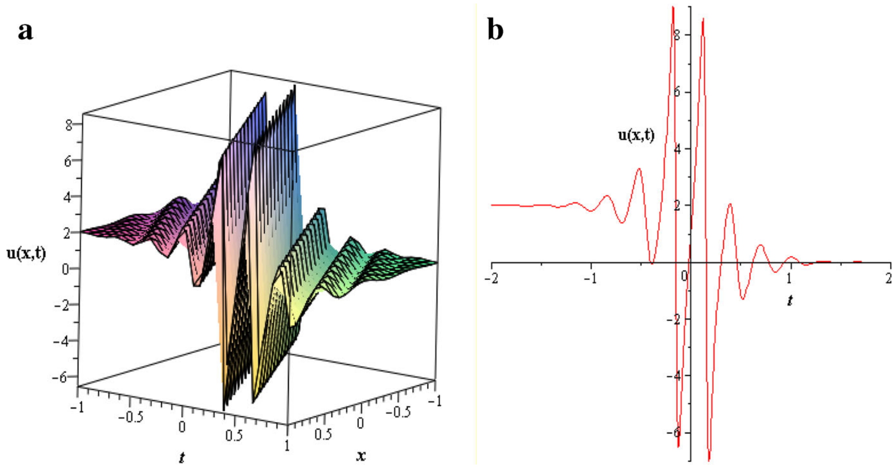

Fig. 2 **a** Kinky-periodic wave solution for $b_0 = 1, k_1 = 1, k_2 = 5$ of the real part of $u_{17,18}$ and **b** 2D plot shows the wave propagation pattern at $x = 0$

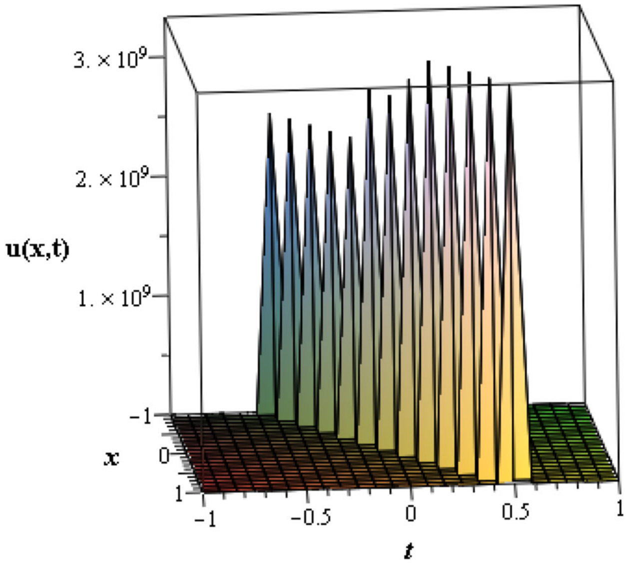

Fig. 3 Multi-soliton solution for $b_0 = 1, k_1 = 1, k_2 = 2$ of the real part of $u_{7,8}$

The complexiton solutions consist with two traveling variables ξ and η expressed in-terms of $\tan\xi$ and $\tanh\eta$, $\sec h\eta$; $\tan\xi$ and $\coth\eta$, $\mathrm{cosec}\, h\eta$ gives bell type-periodic solutions like Fig. 5 of $u_{57,58}$.

Graphs of the Solutions to the Gardner Equation

The complexiton solutions to the Gardner equations consist with two traveling variables ξ and η expressed in-terms of $\tanh\xi$ and $\tan\eta$, $\sec\eta$; $\coth\xi$ and $\tan\eta$, $\sec\eta$; $\tanh\xi$ and $\cot\eta$, $\mathrm{cosec}\,\eta$; $\coth\xi$ and $\tan\eta$, $\sec\eta$ gives the kinky-periodic wave. The Fig. 6 gives this

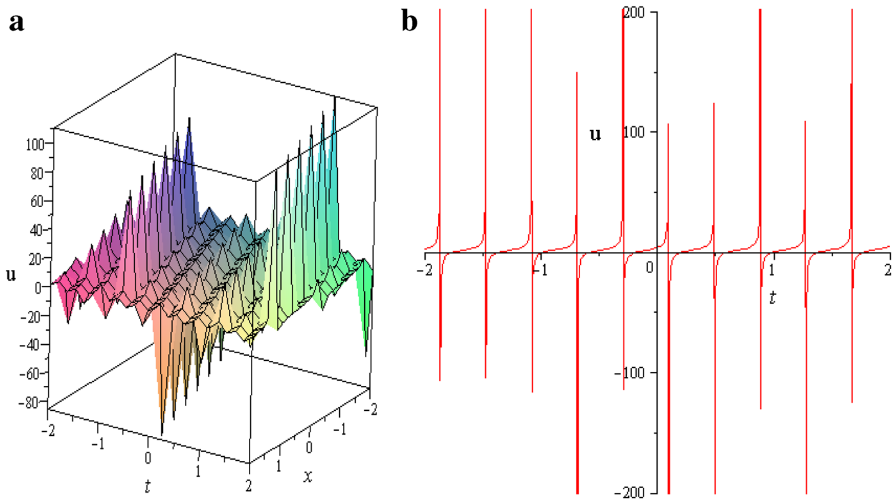

Fig. 4 **a** Doubly-periodic wave solution for $b_0 = 2, k_1 = 1, k_2 = 2$ of $u_{41,42}$. **b** 2D plot shows the wave propagation pattern at $x = 0$

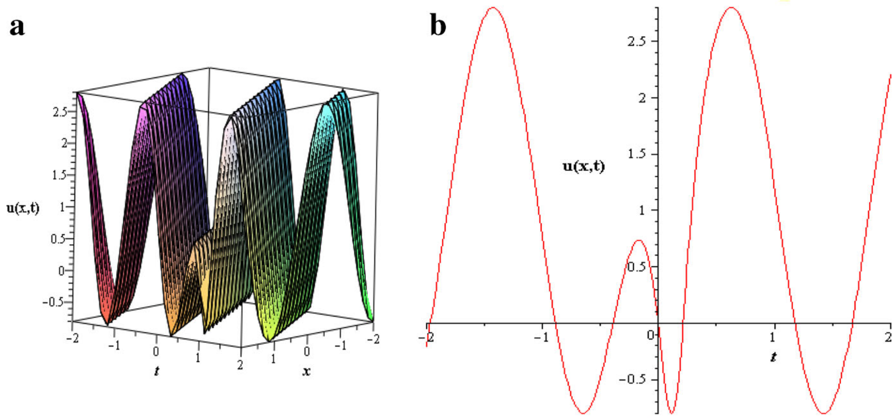

Fig. 5 **a** Bell-periodic wave solution for $b_0 = 1, k_1 = 1, k_2 = 3$ of the real part of $u_{57,58}$. **b** 2D plot shows the wave propagation pattern $x = 0$

type of wave and it is plotted for the solution $u_{1,2}$. The solutions involving combinations of $\tanh\xi$ and $\tan\eta$; $\coth\xi$ and $\cot\eta$ gives kinky-periodic wave solutions like Fig. 7a and it is plotted for the solution $u_{17,18}$. The solutions involving combinations of $\coth\xi$ and $\coth\eta$, $\mathrm{cosec}\,h\eta$; $\tanh\xi$ and $\cot\eta$; $\coth\xi$ and $\tan\eta$; some times $\tanh\xi$ and $\cot\eta$, $\mathrm{cosec}\,\eta$ gives single soliton solutions. The Fig. 8a gives this type of wave and it is plotted for the solutio $u_{11,12}$. The solutions involving combinations of $\tanh\xi$ and $\tanh\eta$, $\sec h\eta$; $\coth\xi$ and $\tanh\eta$, $\sec h\eta$; $\coth\xi$ and $\coth\eta$, $\mathrm{cosec}\,h\eta$ gives collitions of three solitons (two kinks with one bell type wave) solutions like Fig. 8b and it is plotted for the solution $u_{27,28}$.

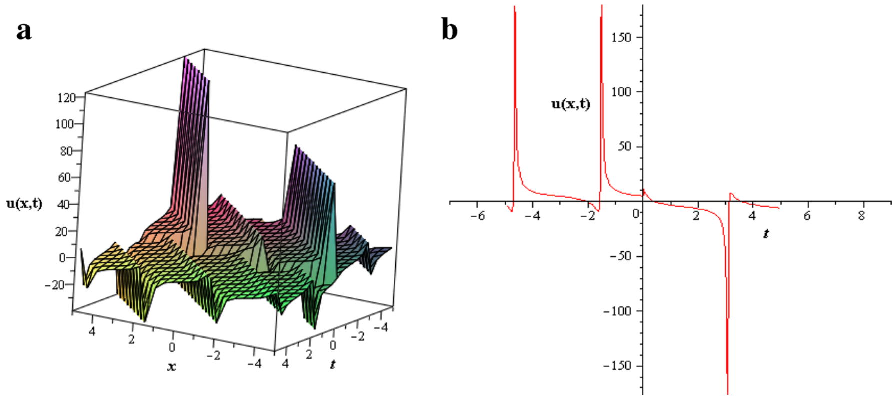

Fig. 6 Cross Kinky-periodic wave solution for $a_4 = 1, b_1 = 0, b_2 = b_3 = w_2 = k_2 = 2, k_1 = 3$ of the real part of $u_{1,2}$ **a** 3D plot and **b** 2D plot along $t = 0$ shows the wave propagation pattern

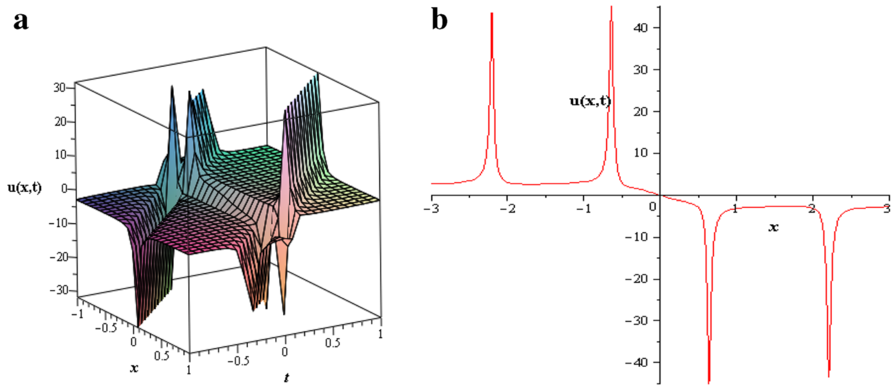

Fig. 7 Kinky-periodic wave solution for $a_4 = 1, b_1 = 0, b_2 = b_3 = w_2 = k_2 = 2, k_1 = 3$ of the real part of $u_{17,18}$ **a** 3D plot and **b** 2D plot along $t = 0$ shows the wave propagation pattern

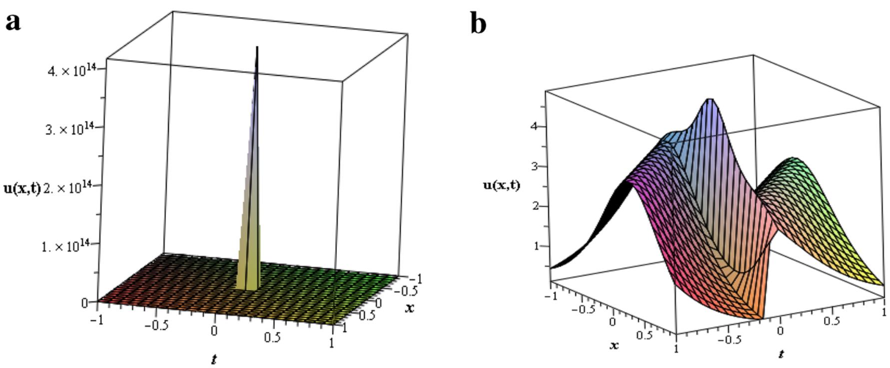

Fig. 8 **a** Single soliton wave solution for $a_4 = 1, b_1 = 0, b_2 = b_3 = w_2 = k_2 = 2, k_1 = 3$ of the real part of $u_{11,12}$ and **b** collision of two kink with a bell shaped soliton solution for $a_4 = 1, b_1 = 0, b_2 = b_3 = w_2 = k_2 = 2, k_1 = 3$ of the real part of $u_{27,28}$

Conclusions

In this paper, modified version of double sub-equation method is proposed for solving nonlinear evolution equation. As a concrete example, we consider the (1 + 1) Dimensional Burger's equation and the $(1+1)$-dimensional Gardner equation (or combined KdV–mKdV). Applying this method, we acquired novel some complexiton solutions in the combination of trigonometric and hyperbolic functions with different structures. It is hoped that the study of these complexiton solutions could further assist understanding, identifying and classifying nonlinear integrable and nonintegrable differential equations and their exact solutions. In fact, we naturally use two or more really different subequations to handle complexiton solution with two different traveling variables i.e., multi-variable Riccati equations. Thus we can obtain more prosperous complexiton solutions possessing a mixture of trigonometric periodic and hyperbolic functions.

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