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Stochastic optimization of level-dependent perishable inventory system by Jackson network

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ABSTRACT

In this research work, we developed a model to investigate the long-run behavior of the solutions of continuous review (s, S) inventory policy for level-dependent perishable products with positive service time on Jackson queueing network. The service facility is assumed that the waiting capacities of each queue are different. In this model, we assume two independent single server nodes in a $M/M/1$ queueing system with an associated inventory, where each node represents a queue. In a Poisson process, customers enter the system. The service times $(\mu_i > 0)$ and lead times $(\theta_i > 0)$ are independent and also spread exponentially at the node i for $1 \leqslant i \leqslant 2$. If the inventory levels of goods from the warehouse i are below s_i at any moment, an order for $Q_i = ((S_i - s_i) > 0; \forall i)$ units is made, where S_i is the highest capacity of storage for each warehouse in the system. In the steady state case, the matrix analytic technique is used to derive the combined probability distribution for the number of customers and inventory level. The overall anticipated cost rate is determined using some essential system performance measures. Numerical and graphical examples are presented to analyze the effect of variation of parameters on the system.

1. Introduction

Jackson Network of the queuing inventory systems of the level dependents perishable items of stock is a very important sophisticated model of the operations research to understand and explain the behavior of the operating systems of real life problems. The investigations of the simulation results of the probability distribution of a number of decision factors such as the rate of arrival, a large quantity of facilities, the service time, the length of the queue, the sorting system, and the waiting time among others of operating parameters of queuing inventory problems play a vital role in various fields of management science, production and industrial engineering. Inventory control of perishable items of stock in the grocery industry is significant challenges for the inventory management. In such situation, a good inventory management system has always a positive effect on customer satisfactions to meet uncertainties in demand. In planning and inventory control of business process management, Jackson network of multi-server queues of different capacity with exponential service times are very much helpful to make optimal decisions of the management systems.

In this work, we explore a continuous review (s, S) stochastic inventory system for level-dependent perishable items with positive service time on an open Jackson network of two single-server for $M/M/1$ type queues. Such stochastic models have wide applications in production and manufacturing engineering systems, telecommunication networks, biological systems, digital cellular mobile systems and also to analyze the service systems. Zhang et al. [1] analyzed the queueing inventory system under the basis of (s, S) policy with server vacations and impatient customers. Islam et al. [2] discussed stochastic perishable inventory system with postponed demands and customers renege at a service facility. Tirkolaee et al. [3] presented a multi-objective model with perishable products for two-step green routing problems.

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Ekramol et al. [4,5] analyzed stochastic perishable inventory system under base stock policy and (s, S) policy on the open Jackson network at a service facility in which there is no priority in serving the consumers. Baron et al. [6] developed a stochastic model for perishable products ordered that arrived in batches with zero lead time. Takine [7] considered a single-server queues whose service speed is determined by the finite state of the underlying Markov chain for several customer classes with Markov-modulated arrivals. Wang and Tai [8] presented an M/M/3 queueing system with finite customer capacity, where each server's service capability are varied and the number of servers varies based on the queue length. Berman and Sapna [9] analyzed a queueing-inventory system with Poisson arrivals, arbitrarily distributed service times and zero lead times at service facilities, where a finite capacity queue of the customers is an M/G/l type. Radhamani et al. [10] considered a discrete-time inventory model where demands arrive according to a discrete Markovian arrival process under (s, S) policy. Yadavalli et al. [11] consider a continuous review catastrophe inventory system in which two substitutable goods are held in disaster management, with no lead time and immediate replenishment. Sivakumar and Arivarignan [12] proposed a perishable inventory model of negative customers at service facility and capacity of waiting hall is infinite. Kouki et al. [13] considered a periodic review (T, S) policy for perishable products with a random lifespan, where demand follows a Poisson trend and procurement lead time is constant. Lawrence et al. [14], Jeganathan et al. [15], Sivakumar et al. [16], Jayaraman et al. [17] discussed a continuous review perishable inventory system where the lifetime of the each item is assumed to have exponential distributions with service facility.

In the above models, the authors assumed that the queuing system had a single service facility with one or more servers for perishable inventory. But, in real life, a lot of communication systems need to be depicted as a network of linked queues for level dependent perishable inventory. A network of queues is a collection of service facilities where customers are serviced at some or all of the facilities. Which is incorporated in this paper by assuming the Jackson network. This is a more robust model than the existing models in the exits literature. Sometimes, customers need more than one particular service, from difference type of service center. For example, healthcare facilities in which level dependent perishable items of pharmaceutical products has the impact on the quality of service, where a patient needing of medical care consisting of more than one diagnosis from cardiology, gastroenterology, radiology, neurology as well as others department of healthcare service centre of the service managements organizations. In this paper our major contribution are summarized as follows: (i) we developed a stochastic model of multi-service facility for queueing inventory systems under the (s, S) optimal policy on the Jackson network for level dependent perishable inventory, (ii) we derived the generator matrix associated with the underlying continuous time Markov chain (CTMC) to find the stochastic behavior of the solutions and (iii) for this model, stability condition is established. The main purpose of this work is to design a control policy for identify an optimal level of service that will minimize the total cost of the system which have a real impact on the ability of the organization to meet future demands of customers. The various operating characteristics are obtained using the matrix analytical method. Sensitivity analysis is provided by the proposed model which is very much important to bring out the qualitative behavior of the queueing inventory system under the variations of various parameters as the model of practical applications.

The remaining part of the paper is carried out as follows: Section 2: assumptions and notations of the model; Section 3: model and analysis, Section 4: stability condition; Section 5: steady-state analysis;Section 6: system performance measures; Section 8: cost analysis; Section 9: to demonstrate the model, numerical examples are presented; Section 9: the optimal solutions are evaluated to a sensitivity analysis with regard to important parameters and finally, the paper is concluded in Section 10.

2. Assumptions and notations

2.1. Assumptions

In this work, we make the following assumptions.

(i) The highest stock level in warehouse i is $S_i; (i = 1, 2)$.
(ii) The lead-time is considered to be exponential distributed with the variable $\theta_i; (i = 1, 2)$.
(iii) the service period of customer base in queue i are mutually independent and exponentially dispersed with the parameter $\mu_i; (i = 1, 2)$,
(iv) Inter-arrival of times of demands of queue i of the customers are exponentially distributed with parameter $\lambda_i; (i = 1, 2)$
(v) When inventory levels of each warehouse is zero, customers no services from the system but customers joints the system.
(vi) The goods are perishable. The lifespan of each item in warehouse i from inventory level j is considered to be exponentially distributed, with a rate of $j\beta_i \ (i = 1, 2; \ j = 1, 2, \ldots, S_i)$, which is level dependant.
(vii) Waiting room of first queue has infinite capacity and second queue has finite capacity.

2.2. Symbols and Notations

This paper makes use of the following symbols and notations.

$[A]_{ij}$	denote $(i,j)^{th}$ element of a matrix A.
$\mathbf{0}$	denote the zero vector.
$\mathbf{e}$	denote $(1, 1, \ldots, 1)^T$ a column vector of $1'$s of appropriate order.
$N_1 (= \infty)$	denote number of infinte customers in first queue.
N_2	denote number of finite customers in second queue.
$\eta_1 =$	$-(\lambda_1 + \lambda_2 + \theta_1 + \theta_2)$.
$\eta_2 =$	$-(\beta_1 + \beta_2 + \mu_2 p_{20} + \mu_2 p_{21})$.
$\eta_3 =$	$-(\mu_1 p_{10} + \mu_1 p_{12})$.

3. Model and Analysis

A Jackson Network is a network made up of a number of nodes, each of which represents a queue with unique service rates for various nodes. In this research, we explore an open Jackson network of two parallel single-server $M/M/1$ queues with a continuous review (s_i, S_i) stochastic level dependent inventory systems for perishable goods. Demands for single units arise according to a Poisson process, where maximum capacity S_i units are used for i th warehouse $(i = 1, 2)$. Queue-1 is a $M/M/1/\infty$ system of infinite capacity single-server queue. The other queue, queue-2, is a $M/M/1/N_2$ system of finite-capacity single-server queue,where N_2 represents the number of consumers.

This network is represented in Fig.1 and satisfies the following properties.

(i) A Poisson process is formed by any external arrivals at node i.
(ii) The service times are exponentially dispersed, and all queues follow the first-come, first-served (FCFS) service discipline.
(iii) the utilization of all of the queues is less than one.

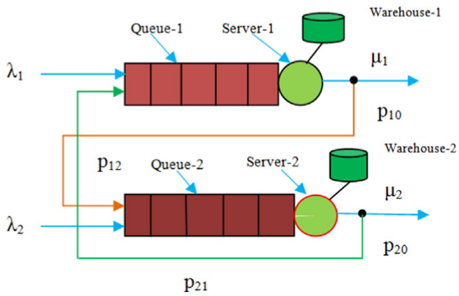

Fig. 1. Schematic diagram of $M/M/1$ type Jackson Network.

(iv) The arrival rate into each queue is λ_i from the outside of the system. The service rate into each queue is μ_i. A customer who has finished service in queue i does have the option of moving to queue j with probability p_{ij} or leaving the system with probability p_{i0}.

$$p_{i0} = 1 - \sum_{j=1}^{2} p_{ij}.$$

Let $I_1(t), I_2(t), J_1(t)$ and $J_2(t)$ be respectively, indicate the amount of customers in the first and second queues, as well as inventory levels in the first and second warehouses, at time t. The assumptions placed on the incoming and outgoing processes may be used to establish that $\{I_1(t), I_2(t), J_1(t), J_2(t) : t \geqslant 0\}$ is a four-dimensional stochastic process with state space.

$$E = \{(x, y, z, w) :\ 0 \leqslant x < \infty,\ 0 \leqslant y \leqslant N_2,\ 0 \leqslant z \leqslant S_1,\ 0 \leqslant w \leqslant S_2\}$$

The infinitesimal generator for this system is defined as,

$$\tilde{A} = (a((x, y, z, w); (m, n, t, r))), (x, y, z, w), (m, n, t, r) \in E$$

and the following parameters can be used to get it:

(i) When a customer arrives at queue x, the state (x, y, z, w) transitions to state $(x+1, y, z, w)$ with intensity λ_1; $x = 0, 1, \ldots, \infty$; $y = 0, 1, \ldots, N_2$; $z = 0, 1, \ldots, S_1$ *and* $w = 0, 1, \ldots, S_2$.

(ii) When a customer arrives at queue y, the state (x, y, z, w) transitions to state $(x, y+1, z, w)$ with intensity λ_2; $x = 0, 1, \ldots, \infty$; $y = 0, 1, \ldots, N_2$; $z = 0, 1, \ldots, S_1$ *and* $w = 0, 1, \ldots, S_2$.

(iii) A customer who completes service at queue x might exit the system, causing the inventory level of the z th warehouse to drop by one item. As a result, a transition from state (x, y, z, w) to state $(x-1, y, z-1, w)$ with intensity $\mu_1 p_{10}$; $x = 1, \ldots, \infty$; $y = 0, 1, \ldots, N_2$; $z = 1, \ldots, S_1$ *and* $w = 0, 1, \ldots, S_2$.

(iv) When a customer completes service at queue y, the system records a one-item reduction in the inventory level of the w th warehouse. As a result, a transition from state (x, y, z, w) to state $(x, y-1, z, w-1)$ with intensity $\mu_2 p_{20}$; $x = 0, 1, \ldots, \infty$; $y = 1, \ldots, N_2$; $z = 0, 1, \ldots, S_1$ *and* $w = 1, \ldots, S_2$.

(v) After finishing service at queue x, a client can transfer to queue y, resulting in a one-item reduction in the inventory level of the z th warehouse. As a result, a transition from state (x, y, z, w) to state $(x-1, y+1, z-1, w)$ with intensity $\mu_1 p_{12}$; $x = 1, \ldots, \infty$; $y = 0, 1, \ldots, N_2$; $z = 1, \ldots, S_1$ *and* $w = 0, 1, \ldots, S_2$.

(vi) After finishing service at queue$y$, a customer can transfer to queue x, resulting in a one-item reduction in the inventory level of the w th warehouse. Thus, a transition with intensity $\mu_2 p_{21}$ from state (x, y, z, w) to state $(x+1, y-1, z, w-1)$; $x = 0, 1, \ldots, \infty$; $y = 1, \ldots, N_2$; $z = 0, 1, \ldots, S_1$ *and* $w = 1, \ldots, S_2$.

(vii) Due to the perishability of a particular item in the z th warehouse. With the intensity $z\beta_1$, a transition occurs from state (x, y, z, w) to state $(x, y, z-1, w)$; $x = 0, 1, \ldots, \infty$; $y = 0, 1, \ldots, N_2$; $z = 1, \ldots, S_1$ *and* $w = 0, 1, \ldots, S_2$.

(viii) For the w th warehouse, due to the perishability of an item. A transition occurs with intensity $w\beta_2$ from state (x, y, z, w) to state $(x, y, z, w-1)$; $x = 0, 1, \ldots, \infty$; $y = 0, 1, \ldots, N_2$; $z = 0, 1, \ldots, S_1$ *and* $w = 1, \ldots, S_2$.

(ix) Due to inventory replenishment in the z th warehouse, a transition from state (x, y, z, w) to state $(x, y, z+Q_1, w)$ occurs with intensity θ_1, where, $Q_1 = S_1 - s_1$; $x = 0, 1, \ldots, \infty$; $y = 0, 1, \ldots, N_2$; $z = 0, 1, \ldots, s_1$ *and* $w = 0, 1, \ldots, S_2$.

(x) A transition occurs from state (x, y, z, w) to state $(x, y, z, w+Q_2)$ with intensityθ_2 due to inventory replenishment at the w th warehouse. where, $Q_2 = S_2 - s_2$; $x = 0, 1, \ldots, \infty$; $y = 0, 1, \ldots, N_2$; $z = 0, 1, \ldots, S_1$ *and* $w = 0, 1, \ldots, s_2$.

(xi) The rate is zero for all other transitions from state (x, y, z, w) to state (m, n, t, r), except when $(x, y, z, w) \neq (m, n, t, r)$.

(xii) Finally, the diagonal entry equals the negative of the total of the other entries in that row. Specifically stated

$$a((x, y, z, w), (m, n, t, r)) = \zeta$$
$$= -\sum_{x}\sum_{y}\sum_{z}\sum_{w}{}_{(x,y,z,w)\neq(m,n,t,r)} a((x, y, z, w), (m, n, t, r))$$

Hence we have

$$\tilde{A} = a((x, y, z, w), (m, n, t, r))$$
$$= \begin{cases}
\lambda_1, & m = x+1, x = 0, 1, \ldots, \infty; \quad n = y, y = 0, 1, \ldots, N_2 \\
 & t = z, k = 0, 1, \ldots, S_1; \quad r = w, w = 0, 1, \ldots, S_2 \\
\lambda_2, & m = x, x = 0, 1, \ldots, \infty; \quad n = y+1, y = 0, 1, \ldots, N_2 \\
 & t = z, z = 0, 1, \ldots, S_1; \quad r = w, w = 0, 1, \ldots, S_2 \\
\mu_1 p_{10}, & m = x-1, x = 1, \ldots, \infty; \quad n = y, y = 0, 1, \ldots, N_2 \\
 & t = z-1, z = 1, \ldots, S_1; \quad r = w, l = 0, 1, \ldots, S_2 \\
\mu_2 p_{20}, & m = x, x = 0, 1, \ldots, \infty; \quad n = y-1, y = 1, \ldots, N_2 \\
 & t = z, z = 0, 1, \ldots, S_1; \quad r = w-1, l = 1, \ldots, S_2 \\
\mu_1 p_{12}, & m = x-1, x = 1, \ldots, \infty; \quad n = y+1, y = 0, 1, \ldots, N_2 \\
 & t = z-1, z = 1, \ldots, S_1; \quad r = w, w = 0, 1, \ldots, S_2 \\
\mu_2 p_{21}, & m = x+1, x = 0, 1, \ldots, \infty; \quad n = y-1, y = 1, \ldots, N_2 \\
 & t = z, z = 0, 1, \ldots, S_1; \quad r = w-1, w = 1, \ldots, S_2 \\
z\beta_1, & m = x, x = 0, 1, \ldots, \infty; \quad n = y, y = 0, 1, \ldots, N_2 \\
 & t = z-1, z = 1, \ldots, S_1; \quad r = w, w = 0, 1, \ldots, S_2 \\
w\beta_2, & m = x, x = 0, 1, \ldots, \infty; \quad n = y, y = 0, 1, \ldots, N_2 \\
 & t = z, z = 0, 1, \ldots, S_1; \quad r = w-1, w = 1, \ldots, S_2 \\
\theta_1, & m = x, x = 0, 1, \ldots, \infty; \quad n = y, y = 0, 1, \ldots, N_2 \\
 & t = z+Q_1, z = 0, 1, \ldots, s_1; \quad r = w, w = 0, 1, \ldots, S_2 \\
\theta_2, & m = x, x = 0, 1, \ldots, \infty; \quad n = y, y = 0, 1, \ldots, N_2 \\
 & t = z, z = 0, 1, \ldots, S_1; \quad r = w+Q_2, w = 0, 1, \ldots, s_2 \\
\zeta, & m = x, x = 0, 1, \ldots, \infty; \quad n = y, y = 0, 1, \ldots, N_2 \\
 & t = z, z = 0, 1, \ldots, S_1; \quad r = w, w = 0, 1, \ldots, S_2 \\
0, & \text{otherwise}
\end{cases}$$

The infinitesimal generator Matrix $\tilde{A}$ of the process has the following block tridiagonal matrix structure:

$$\tilde{A} = \begin{bmatrix} B_0^{(0)} & A_0^{(0)} & 0 & 0 & \ldots \\ A_2^{(1)} & A_1^{(1)} & A_0^{(1)} & 0 & \ldots \\ 0 & A_2^{(2)} & A_1^{(2)} & A_0^{(2)} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where, the blocks $B_0^{(0)}$ and for $x \geqslant 0$ and $l = 0,1,2;$ $A_l^{(x)}$ are squares matrices,each order is $((Q_1+N_2)(Q_1+S_2N_2)) \times ((Q_1+N_2)(Q_1+S_2N_2))$, $B_0^{(i)}$ depicts the transition within the level x, $x \geqslant 0$, $A_0^{(0)}$ represents the transition from level x to $x+1$, $x \geqslant 0$, $A_2^{(x)}$ illustrates the transition from level x to $x-1$, $x \geqslant 1$, $A_1^{(x)}$ shows the transition within the level $x \geqslant 1$. So the infinitesimal generator Matrix $\widetilde{A}$ may be written as a partitioned matrix

$$\widetilde{A} = (A_{xm}) = (a((x,y,z,w);(m,n,t,r))), (x,y,z,w), (m,n,t,r) \in E$$

which is given by

$$\widetilde{A} = (A_{xm}) = \begin{cases} B_0^{(x)}; & m = x \quad x = 0 \\ A_0^{(x)}; & m = x+1 \quad x = 0,1,2,\ldots \\ A_1^{(x)}; & m = x \quad x = 1,2,\ldots \\ A_2^{(x)}; & m = x-1 \quad x = 1,2,\ldots \\ \mathbf{0} & \text{otherwise} \end{cases}$$

where B_0 is a $(s_1+1) \times (s_2+1)$ sub-matrix which are given by

$$B_0 = \left[B_0^{(0)}\right]_{xy} = \begin{cases} B_1; & y = x \quad x = s_1 \\ B_2; & y = x+1 \quad x = s_1 \\ B_3; & y = x-1 \quad x = s_1+1 \\ B_4; & y = x \quad x = s_1+1 \end{cases}$$

where B_1, B_2, B_3, and B_4 are square sub matrices of each order is $(Q_1(S_1+N_2)+Q_1S_2) \times (Q_1(S_1+N_2)+Q_1S_2)$ respectively.

$$B_1 = \begin{cases} z\beta_1, & m=x, x=0 \quad n=y, y=0 \\ & t=z-1, z=1,\ldots,S_1; \quad r=w, w=0,1,\ldots,S_2. \\ w\beta_2, & m=x, x=0 \quad n=y, y=0 \\ & t=z, z=0,1,\ldots,S_1; \quad r=w-1, w=1,\ldots,S_2. \\ \theta_1, & m=x, x=0 \quad n=y, y=0 \\ & t=z+Q_1, z=0,1; \quad r=w, w=0,1,\ldots,S_2. \\ \theta_2, & m=x, x=0 \quad n=y, y=0 \\ & t=z, z=0,1,\ldots,S_1; \quad r=w+Q_2, w=0,1 \\ \eta_1, & m=x, x=0 \quad n=y, y=0 \\ & t=z, z=s_1-1; \quad r=w, w=s_2-1 \\ \eta_1 - w\beta_2, & m=x, x=0 \quad n=y, y=0 \\ & t=z, z=s_1-1; \quad r=w, w=s_2 \\ \eta_1+\theta_2-w\beta_2, & m=x, x=0 \quad n=y, y=0 \\ & t=z, z=s_1-1; \quad r=w, w=s_2+1, S_2 \\ \eta_1 - z\beta_1, & m=x, x=0 \quad n=y, y=0 \\ & t=z, z=s_1; \quad r=w, w=s_2-1 \\ \eta_1 - z\beta_1 - w\beta_2, & m=x, x=0 \quad n=y, y=0 \\ & t=z, z=s_1; \quad r=w, w=s_2 \\ \eta_1 - z\beta_1 - w\beta_2 + \theta_2, & m=x, x=0 \quad n=y, y=0 \\ & t=z, z=s_1; \quad r=w, w=s_2+1, S_2 \\ \eta_1+\theta_1-z\beta_1, & m=x, x=0 \quad n=y, y=0 \\ & t=z, z=s_1+1, S_1; \quad r=w, w=s_2-1 \\ \eta_1+\theta_1-z\beta_1-w\beta_2, & m=x, x=0 \quad n=y, y=0 \\ & t=z, z=s_1+1, S_1; \quad r=w, w=s_2 \\ -(z\beta_1+w\beta_2+\lambda_1+\lambda_2), & m=x, x=0 \quad n=y, y=0 \\ & t=z, z=s_1+1, S_1; \quad r=w, w=s_2+1, S_2 \\ 0; & \text{otherwise} \end{cases}$$

$$B_2 = \begin{cases} \lambda_2, & m = x, \ x = 0 \quad n = y+1, \ y = 0 \\ & t = z, \ z = 0,1,\ldots,S_1; \quad r = w, \ l = 0,1,\ldots,S_2. \\ 0; & \text{otherwise} \end{cases}$$

$$B_3 = \begin{cases} \mu_2 p_{20}, & m = x, \ x = 0 \quad n = y-1, \ y = 1 \\ & t = z, \ z = 0,1,\ldots,S_1; \quad r = w-1, \ w = 1,\ldots,S_2 \\ 0; & \text{otherwise} \end{cases}$$

$$B_4 = \begin{cases} z\beta_1, & m=x, x=0 \quad n=y, y=1 \\ & t=z-1, z=1,\ldots,S_1; \quad r=w, w=0,1,\ldots,S_2 \\ w\beta_2, & m=x, x=0 \quad n=y, y=1 \\ & t=z-1, z=0,1,\ldots,S_1; \quad r=w-1, w=1,\ldots,S_2 \\ \theta_1, & m=x, x=0 \quad n=y, y=1 \\ & t=z+Q_1, z=s_1-1, s_1; \quad r=w, w=0,1,\ldots,S_2 \\ \theta_2, & m=x, x=0 \quad n=y, y=1 \\ & t=z, z=0,1,\ldots,S_1; \quad r=w+Q_2, w=s_2-1, s_2 \\ \eta_1+\lambda_2, & m=x, x=0 \quad n=y, y=1 \\ & t=z, z=s_1-1; \quad r=w, w=s_2-1 \\ \eta_1+\lambda_2+z\beta_1+\eta_2, & m=x, x=0 \quad n=y, y=1 \\ & t=z, z=s_1-1; \quad r=w, w=s_2 \\ z\beta_1+\eta_1-\theta_1-\lambda_1, & m=x, x=0 \quad n=y, y=1 \\ & t=z, z=s_1-1; \quad r=w, w=s_2+1, S_2 \\ \eta_1-z\beta_1+\lambda_2, & m=x, x=0 \quad n=y, y=1 \\ & t=z, z=s_1; \quad r=w, w=s_2-1 \\ \eta_1+\eta_2+\lambda_2, & m=x, x=0 \quad n=y, y=1 \\ & t=z, z=s_1; \quad r=w, w=s_2 \\ \eta_2-\lambda_1-\theta_1, & m=x, x=0 \quad n=y, y=1 \\ & t=z, z=s_1; \quad r=w, w=s_2+1, S_2 \\ -(\lambda_1+z\beta_1+\theta_2), & m=x, x=0 \quad n=y, y=1 \\ & t=z, z=s_1+1, S_2,; \quad r=w, w=s_2-1 \\ \eta_2-\lambda_1-\theta_2, & m=xi, x=0 \quad n=y, y=1 \\ & t=z, z=s_1+1, S_2,; \quad r=w, w=s_2 \\ -\lambda_1+\eta_2, & m=x, x=0 \quad n=y, y=1 \\ & t=z, z=s_1+1, S_1,; \quad r=w, w=s_2+1, S_2 \\ 0; & \text{otherwise} \end{cases}$$

for level $x = 1,2,\ldots;$

$$A_2 = A_2^{(x)} = \begin{cases} \mu_1 p_{10}, & m=x-1, \quad n=y, y=0,1,\ldots,N_2 \\ & t=z-1, z=1,\ldots,S_1; \quad r=w, w=0,1,\ldots,S_2. \\ \mu_1 p_{12}, & m=x-1, \quad n=y+1, y=0,1,\ldots,N_2 \\ & t=z-1, z=1,\ldots,S_1; \quad r=w, w=0,1,\ldots,S_2. \\ 0; & \text{otherwise} \end{cases}$$

for level $x = 0,1,2,\ldots;$

$$A_0 = A_0^{(x)} = \begin{cases} \lambda_1, & m=x+1, \quad n=y, y=0,1,\ldots,N_2 \\ & t=z, z=0,1,\ldots,S_1; \quad r=w, w=0,1,\ldots,S_2. \\ \mu_2 p_{21}, & m=x+1, \quad n=y-1, y=1,\ldots,N_2 \\ & t=z, z=0,1,\ldots,S_1; \quad r=w-1, w=1,\ldots,S_2. \\ 0; & \text{otherwise} \end{cases}$$

for the level $x = 1,2,\ldots;$

$$A_1 = \left[A_1^{(x)}\right]_{yz} = \begin{cases} A_{11}; & z = y \quad y = s_1 \\ A_{12}; & z = y+1 \quad y = s_1 \\ A_{21}; & z = y-1 \quad y = s_1+1 \\ A_{22}; & z = y \quad y = s_1+1 \\ \mathbf{0} & \text{otherwise} \end{cases}$$

where A_{11}, A_{12}, A_{21}, and A_{22} are square sub matrices of each order is $(N_2+Q_1)(S_1+1) \times (N_2+Q_1)(S_1+1)$ respectively.

$$A_{11}=\begin{cases} z\beta_1, & m=x,x=1 & n=y,y=0 \\ & t=z-1,z=1,\ldots,S_1; & r=w,w=0,1,\ldots,S_2. \\ w\beta_2, & m=x,x=1 & n=y,y=0 \\ & t=z,z=0,1,\ldots,S_1; & r=w-1,w=1,\ldots,S_2. \\ \theta_1, & m=x,x=1 & n=y,y=0 \\ & t=z+Q_1,z=0,1; & r=w,w=0,1,\ldots,S_2. \\ \theta_2, & m=x,x=1 & n=y,y=0 \\ & t=z,z=0,1,\ldots,S_1; & r=w+Q_2,w=0,1 \\ \eta_1, & m=x,x=1 & n=z,z=0 \\ & t=z,z=s_1-1; & r=w,w=s_2-1 \\ \eta_1-w\beta_2, & m=x,x=1 & n=y,y=0 \\ & t=z,z=s_1-1; & r=w,w=s_2 \\ \eta_1+\theta_2-w\beta_2, & m=x,x=1 & n=y,y=0 \\ & t=z,z=s_1-1; & r=w,w=s_2+1,S_2 \\ \eta_1-z\beta_1+\eta_3, & m=x,x=1 & n=y,y=0 \\ & t=z,z=s_1; & r=w,w=s_2-1 \\ \eta_1-z\beta_1-w\beta_2+\eta_3, & m=x,x=1 & n=y,y=0 \\ & t=z,z=s_1; & r=w,w=s_2 \\ \eta_1-z\beta_1-w\beta_2+\theta_2+\eta_3, & m=x,x=1 & n=y,y=0 \\ & t=z,z=s_1; & r=w,w=s_2+1,S_2 \\ \eta_1+\theta_1-z\beta_1+\eta_3, & m=x,x=1 & n=y,y=0 \\ & t=z,z=s_1+1,S_1; & r=w,w=s_2-1 \\ \eta_1+\theta_1-z\beta_1-w\beta_2+\eta_3, & m=x,x=1 & n=y,y=0 \\ & t=y,y=s_1+1,S_1; & r=w,w=s_2 \\ -(z\beta_1+w\beta_2+\lambda_1+\lambda_2)+\eta_3, & m=x,x=1 & n=y,y=0 \\ & t=z,z=s_1+1,S_1; & r=w,w=s_2+1,S_2 \\ 0; & \text{otherwise} & \end{cases}$$

$$A_{12}=\begin{cases} \lambda_2, & m=x,\ x=0, & n=y+1,\ y=0 \\ & t=z,\ z=0,1,\ldots,S_1; & r=w,\ w=0,1,\ldots,S_2 \\ 0; & \text{otherwise} & \end{cases}$$

$$A_{21}=\begin{cases} \mu_2p_{20}, & m=x,\ x=0, & n=y-1,\ y=1 \\ & t=z,\ z=0,1,\ldots,S_1; & r=w,\ w=1,\ldots,S_2 \\ 0; & \text{otherwise} & \end{cases}$$

$$A_{22}=\begin{cases} z\beta_1, & m=x,x=1 & n=y,y=1 \\ & t=z-1,z=1,\ldots,S_1; & r=w,w=0,1,\ldots,S_2 \\ w\beta_2, & m=x,x=1 & n=y,y=1 \\ & t=z,z=0,1,\ldots,S_1; & r=w-1,w=1,\ldots,S_2 \\ \theta_1, & m=x,x=1 & n=y,y=1 \\ & t=z+Q_1,z=s_1-1,s_1; & r=w,w=0,1,\ldots,S_2 \\ \theta_2, & m=x, & n=y,y=1 \\ & t=z,z=0,1,\ldots,S_1; & r=w+Q_2,wl=s_2-1,s_2 \\ \eta_1+\lambda_2, & m=x,x=1 & n=y,y=1 \\ & t=z,z=s_1-1; & r=w,w=s_2-1 \\ \eta_1+\lambda_2+z\beta_1+\eta_2, & m=x,x=1 & n=y,y=1 \\ & t=z,z=s_1-1; & r=w,w=s_2 \\ z\beta_1+\eta_1-\theta_1-\lambda_1+\eta_2, & m=x,x=1 & n=y,y=1 \\ & t=z,z=s_1-1; & r=w,w=s_2+1,S_2 \\ \eta_1-z\beta_1+\lambda_2-\mu_1p_{10}, & m=x,x=1 & n=y,y=1 \\ & t=z,z=s_1; & r=w,w=s_2-1 \\ \eta_1+\eta_2+\lambda_2-\mu_1p_{10}, & m=x,x=1 & n=y,y=1 \\ & t=z,z=s_1; & r=w,w=s_2 \\ \eta_2-\lambda_1-\theta_1-\mu_1p_{10}, & m=x,x=1 & n=y,y=1 \\ & t=z,z=s_1; & r=w,w=s_2+1,S_2 \\ -(\lambda_1+z\beta_1+\theta_2+\mu_1p_{10}), & m=x,x=1 & n=y,y=1 \\ & t=z,z=s_1+1,S_2,; & r=w,w=s_2-1 \\ \eta_2-\lambda_1-\theta_2-\mu_1p_{10}, & m=x,x=1 & n=y,y=1 \\ & t=z,z=s_1+1,S_2,; & r=w,w=s_2 \\ -\lambda_1+\eta_2+\eta_2, & m=x,x=1 & n=y,y=1 \\ & t=z,z=s_1+1,S_1,; & r=w,w=s_2+1,S_2 \\ 0; & \text{otherwise} & \end{cases}$$

4. Stability Codition

Let $A=A_0+A_1+A_2$ be the generator matrix of steady-state probability vector π. That is, π fulfills the conditions π satisfies $\pi A=\mathbf{0}$ and $\pi e=1$, where e is a column vector of 1's in the proper order. The markov chain under discussion is a quasi-birth–death process with a level dependence. The queueing system is stable if and only if

$$\pi A_0 e<\pi A_2 e \tag{4.1}$$

according to the QBD structure of the model.

Using (4.1), we may determine the stability condition of the queueing system under consideration. we have,

$$\begin{aligned}&\lambda_1\sum_{x=0}^{Q_1-2}\sum_{y=0}^{Q_2-1}\sum_{z=0}^{S_1}\sum_{w=0}^{S_2}\pi(x,y,z,w)\\ &\quad+\mu_2p_{21}\sum_{x=0}^{Q_1-2}\sum_{y=1}^{Q_2-1}\sum_{z=0}^{S_1}\sum_{w=1}^{S_2}\pi(x,y,z,w)\\ &\quad<\mu_1p_{10}\sum_{x=1}^{Q_1-1}\sum_{y=0}^{Q_2-1}\sum_{z=1}^{S_1}\sum_{w=0}^{S_2}\pi(x,y,z,w)\\ &\qquad+\mu_1p_{12}\sum_{x=1}^{Q_1-1}\sum_{y=0}^{Q_2-2}\sum_{z=1}^{S_1}\sum_{w=0}^{S_2}\pi(x,y,z,w)\end{aligned} \tag{4.2}$$

5. Steady-State Analysis

The steady state probability vector of $\widetilde{A}$ under the stability criterion is determined in this section. The structure of the rate matrix $\widetilde{A}$ shows that the homogeneous Markov process $\{I_1(t),I_2(t),J_1(t),J_2(t):t\geqslant 0\}$on the state space E is irreducible. Hence, the limiting probability of the Stochastic Process

$$\pi(x,y,z,w)=\lim_{t\to\infty}Pr[I_1(t)=x,I_2(t)=y,J_1(t)=z,$$
$$J_2(t)=w\mid I_1(0),I_2(0),J_1(0),J_2(0)]$$

exists.

The probability vector π in steady state, partitioned into levels as follows:

$$\pi=(\pi(0),\pi(1),\pi(2),\ldots,\pi(N_1-1),\pi(N_1),\ldots)$$

That is, π satisfies the conditions:

$$\pi\widetilde{A}=0\,\text{and}\quad\sum_{x=0}^{N_1=\infty}\sum_{y=0}^{N_2}\sum_{z=0}^{S_1}\sum_{w=0}^{S_2}\pi(x,y,z,w)=1$$

where the subvectors $\pi(x),x\geqslant 0$ contains $((2N+1)(S_1+S_2)+(s_1+s_1))$ elements and let

$$\pi(x,y,z)=\pi(x,y,z,w);0\leqslant w\leqslant S_2$$

$$\pi(x,y)=(\pi(x,y,0),\pi(x,y,1),\ldots,\pi(x,y,z));0\leqslant z\leqslant S_1$$

$$\pi(x)=(\pi(x,0),\pi(x,1),\ldots,\pi(x,y));0\leqslant y\leqslant N_2$$

$$\pi=(\pi(0),\pi(1),\ldots,\pi(x));0\leqslant x<\infty$$

The sub-vectors satisfies the equations

$$\pi(0)B_0+\pi(1)A_2=0 \tag{5.1}$$
$$\pi(x-1)A_0+\pi(x)A_1+\pi(x+1)A_2=0;x\geqslant 1. \tag{5.2}$$

The steady state probability vectors π are provided by when the stability requirement (4.2) is satisfied.

$$\pi(x)=\pi(0)R^x;x\geqslant 1.$$

where R is a matrix that satisfies the quadratic matrix problem.

$R^2A_2 + RA_1 + A_0 = 0$

By solving the following equation, the vector $\pi(0)$ is obtained.

$\pi(0)[B_0 + RA_2] = 0$

condition of normalization

$$\sum_{k=0}^{\infty}\pi(k)e = \sum_{k=0}^{\infty}\pi(0)R^k e = 1$$

$i.e.,\ \pi(0)(I-R)^{-1}e = 1$

Proof: The theorem derives from (Neuts[18]) well-known work on matrix-geometric techniques.

6. System Performance Measures

We can determine several essential system performance measures after computing the steady state probability vector. To bring forth the qualitative feature of the model under investigation, we develop several stationary performance indicators of the system in this part. We can compute the overall expected cost per unit time using these indicators.

(i) The average number of customers in the system is specified by α_1 and is provided by,

$$\alpha_1 = \sum_{x=1}^{\infty}\sum_{y=0}^{N_2}\sum_{z=0}^{S_1}\sum_{w=0}^{S_2} x\pi(x,y,z,w)e + \sum_{x=0}^{\infty}\sum_{y=1}^{N_2}\sum_{z=0}^{S_1}\sum_{w=0}^{S_2} y\pi(x,y,z,w)e$$

(ii) The expected inventory level of the system is represented by α_2, which is given by,

$$\alpha_2 = \sum_{x=0}^{\infty}\sum_{y=0}^{N_2}\sum_{z=1}^{S_1}\sum_{w=0}^{S_2} z\pi(x,y,z,w)e + \sum_{x=0}^{\infty}\sum_{y=0}^{N_2}\sum_{z=0}^{S_1}\sum_{w=1}^{S_2} w\pi(x,y,z,w)e$$

(iii) Expected replenishment rate in the system is denoted by α_3 and is given by,

$$\alpha_3 = \theta_1\sum_{x=0}^{\infty}\sum_{y=0}^{N_2}\sum_{z=0}^{s_1}\sum_{w=0}^{S_2}\pi(x,y,z,w)e + \theta_2\sum_{x=0}^{\infty}\sum_{y=0}^{N_2}\sum_{z=0}^{S_1}\sum_{w=0}^{s_2}\pi(x,y,z,w)e$$

(iv) Expected Perishable rate in the system is denoted by α_4 and is given by,

$$\alpha_4 = \sum_{x=0}^{\infty}\sum_{y=0}^{N_2}\sum_{z=1}^{S_1}\sum_{w=0}^{S_2}(z\beta_1)\pi(x,y,z,w)e + \sum_{x=0}^{\infty}\sum_{y=0}^{N_2}\sum_{z=0}^{S_1}\sum_{w=1}^{S_2}(w\beta_2)\pi(x,y,z,w)e$$

(v) The average number of departures in the system after ending service is indicated by α_5 and is provided by,

$$\alpha_5 = \mu_1 p_{10}\sum_{x=1}^{\infty}\sum_{y=0}^{N_2}\sum_{z=1}^{S_1}\sum_{w=0}^{S_2}\pi(x,y,z,w)e + \mu_2 p_{20}\sum_{x=0}^{\infty}\sum_{y=1}^{N_2}\sum_{z=0}^{S_1}\sum_{w=1}^{S_2}\pi(x,y,z,w)e$$

(vi) Expected reorder rate in the system is denoted by α_6 and is defined by,

$$\alpha_6 = (\mu_1 p_{12} + \mu_1 p_{10} + (z+s_1)\beta_1)\sum_{x=1}^{\infty}\sum_{y=0}^{N_2}\sum_{z=1}^{s_1}\sum_{w=0}^{S_2}\pi(x,y,z+s_1,l)e + (\mu_2 p_{21} + \mu_2 p_{20} + (w+s_2)\beta_2)\sum_{x=0}^{\infty}\sum_{y=1}^{N_2}\sum_{z=0}^{S_1}\sum_{w=1}^{s_2}\pi(x,y,z,w+s_2)e$$

(vii) Expected waiting time of a customer in the system is denoted by α_7 and is defined by,

$$\alpha_7 = \frac{\alpha_1}{\eta}; \quad \eta = \sum_{i=1}^{2}\lambda_i$$

7. Cost Analysis

The following costs are taken into account when calculating the expected total cost per unit of time. where,

c_1= Cost of a customer's waiting time in the system per unit of time.

c_2= The inventory carrying cost per unit item in the system per unit time.

c_3= cost of replenishing an item per unit of time in the system.

c_4= Cost per perishable item per unit time in the system.

c_5= cost of customer service per unit of time spent in the system.

c_6= setup cost of per order.

We introduce the cost function in this model, which is defined as the system's long-run anticipated total cost (ETC) rate, which is provided by

$ETC = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4 + c_5\alpha_5 + c_6\alpha_6.$

It's difficult to demonstrate the convexity of the total predicted costs since the π's are computed in a recursive manner. However, in many situations, it may be feasible to establish the computability of the result and the presence of local optima by treating the cost as a function of only two variables, which is plainly a limited case and therefore avoided. Naturally, some of the system's most essential aspects may be explored, as well as a sensitivity analysis.

8. Numerical Illustration

In this part, we fix the parameter values involved in the system to offer measurements of system performance. we vary over the parameters $\lambda_1, \lambda_2, \mu_1, \mu_2, \beta_1, \beta_2, \theta_1$ and θ_2. The values of the system performance metrics are presented for different values of these parameters. Towards this end, we first fixed the parameters values as $S_1 = S_2 = 3, s_1 = s_2 = 1, Q_1 = 2, Q_2 = 2, N_2 = 2, p_{10} = 0.6,\ p_{20} = 0.5, p_{12} = 0.4, p_{21} = 0.5$. Tables 1 through 4 show the numerical results, we observed the following:

- We observed that the results which is presented in Table 1, by fixing the parameters $\mu_1 = 3.0,\ \mu_2 = 2,\ \beta_1 = 0.07,\ \beta_2 = 0.08,\ \theta_1 = 2,\ \theta_2 = 2$. As demand rates λ_1 and λ_2 increases in the system,the stockout occurs frequently. Hence the optimal values of performance measures $\alpha_1, \alpha_3, \alpha_5, \alpha_6, \alpha_7$ are increasing and α_2, α_4 are decreasing because of service facilities for a lot of consumers in the system.
- We illustrated that the results which is provided in Table 2, by fixing the parameters $\lambda_1 = 0.02, \lambda_2 = 0.03, \beta_1 = 0.07; \beta_2 = 0.08; \theta_1 = 2; \theta_2 = 2$.When service rates μ_1 and μ_2 are increases in the system, as is to be predicted., then the optimum results of performance measures $\alpha_2, \alpha_3, \alpha_4, \alpha_5$are increasing and also

Table 1
Effect of arrival rates λ_1 and λ_2 on some performance measures.

(λ_1, λ_2)	α_1	α_2	α_3	α_4	α_5	α_6	α_7
(0.02, 0.03)	0.0265721	4.17617	0.307396	0.311082	0.0907346	0.11490	0.531442
(0.03, 0.04)	0.0387259	4.15165	0.335325	0.309225	0.126732	0.116323	0.553228
(0.04, 0.05)	0.0510531	4.12691	0.363104	0.307355	0.162703	0.118341	0.567257
(0.05, 0.06)	0.063561	4.10206	0.390742	0.305479	0.198651	0.120934	0.577827
(0.06, 0.07)	0.0762568	4.07716	0.418248	0.303602	0.23458	0.124084	0.586591
(0.07, 0.08)	0.0891481	4.05227	0.44563	0.301727	0.270494	0.127774	0.59432
(0.08, 0.09)	0.102242	4.02742	0.472892	0.299859	0.306397	0.131989	0.601426
(0.09, 0.10)	0.115548	4.00267	0.500043	0.297999	0.342292	0.136715	0.608146

Table 2
Effect of service rates μ_1 and μ_2 on some performance measures.

(μ_1, μ_2)	α_1	α_2	α_3	α_4	α_5	α_6	α_7
(3.00, 2.00)	0.0265721	4.17617	0.307396	0.311082	0.0907346	0.114900	0.531442
(3.10, 2.10)	0.0258128	4.17944	0.307398	0.311327	0.0907877	0.114852	0.516256
(3.20, 2.20)	0.025097	4.18237	0.307394	0.311546	0.0908356	0.114807	0.501939
(3.30, 2.30)	0.0244212	4.18500	0.307387	0.311742	0.0908790	0.114765	0.488423
(3.40, 2.40)	0.0237823	4.18736	0.307377	0.311919	0.0909185	0.114725	0.475645
(3.50, 2.50)	0.0231774	4.18951	0.307364	0.312079	0.0909546	0.114687	0.463548
(3.60, 2.60)	0.0226041	4.19145	0.307349	0.312225	0.0909877	0.114651	0.452081
(3.70, 2.70)	0.0220599	4.19322	0.307333	0.312357	0.0910181	0.114617	0.441198

Table 3
Effect of perishable rates β_1 and β_2 on some performance measures.

(β_1, β_2)	α_1	α_2	α_3	α_4	α_5	α_6	α_7
(0.01, 0.02)	0.0263531	4.33248	0.123668	0.0626785	0.0907428	0.0264868	0.527061
(0.02, 0.03)	0.0263826	4.29574	0.155115	0.105164	0.090744	0.0416996	0.527652
(0.03, 0.04)	0.0264143	4.26727	0.186206	0.147165	0.0907433	0.056763	0.528287
(0.04, 0.05)	0.026449	4.24234	0.216971	0.18874	0.0907418	0.0716261	0.52898
(0.05, 0.06)	0.0264869	4.21928	0.247421	0.229911	0.0907397	0.086273	0.529737
(0.06, 0.07)	0.0265279	4.19735	0.277562	0.270689	0.0907373	0.100698	0.530558
(0.07, 0.08)	0.0265721	4.17617	0.307396	0.311082	0.0907346	0.114900	0.531442
(0.08, 0.09)	0.0266195	4.15552	0.336926	0.351096	0.0907317	0.128879	0.532389

Table 4
Effect of replenishment rate θ_1 and θ_2 on some performance measures.

(θ_1, θ_2)	α_1	α_2	α_3	α_4	α_5	α_6	α_7
(2.01, 2.02)	0.026569	4.17765	0.307518	0.311196	0.0907334	0.114974	0.531381
(2.03, 2.04)	0.0265639	4.17857	0.307583	0.311266	0.0907318	0.115010	0.531279
(2.05, 2.06)	0.0265541	4.18038	0.307711	0.311402	0.0907285	0.115082	0.531081
(2.07, 2.08)	0.0265445	4.18216	0.307837	0.311536	0.0907254	0.115151	0.530891
(2.09, 2.10)	0.0265399	4.18303	0.307899	0.311602	0.0907238	0.115186	0.530798
(2.11, 2.12)	0.0265223	4.18645	0.308143	0.311859	0.0907177	0.115320	0.530446
(2.13, 2.14)	0.0265139	4.18811	0.308262	0.311984	0.0907148	0.115385	0.530279
(2.15, 2.16)	0.0265059	4.18974	0.308379	0.312107	0.0907118	0.115450	0.530118

$\alpha_1, \alpha_6, \alpha_7$are decreasing because of providing quality services or an adequate number of servers to minimize the customers waiting time.

- From Table 3, we observed that the numerical results by fixing the parameters $\lambda_1 = 0.02, \lambda_2 = 0.03, \mu_1 = 3.0; \mu_2 = 2.0;\ \theta_1 = 2;$ $\theta_2 = 2$. As is to be expected as perishable rates β_1 and β_2 increases in the system, the inventory level becomes decreases frequently, then $\alpha_1, \alpha_3, \alpha_4, \alpha_6, \alpha_7$are increasing and $\alpha_2, \alpha_5$is decreasing because more perishable items require adequate handling and preservation under an inventory control strategy to provide effective customer service by employing products from a stock.
- From Table 4, we observed that the calculate results by fixing the parameters $\lambda_1 = 0.02; \lambda_2 = 0.03; \mu_1 = 3.0; \mu_2 = 2.0;\ \beta_1 = 0.07;$ $\beta_2 = 0.08$. If replenishment rates θ_1 and θ_2 increases in the system, as is to be expected, the optimal reorder level and highest stock levels grow to avoid a loss of demand in the service, then $\alpha_1, \alpha_5, \alpha_7$ decreases and also $\alpha_2, \alpha_3, \alpha_4, \alpha_6$increases because of stocks of inventory items which are available for fulfillment of a customer demand.

9. Sensitivity Analysis

We investigate the influence of the demand rates λ_1 and λ_2, the perishable rates β_1 and β_2, services rates μ_1 and μ_2 for server-1 and server-2, respectively, based on the overall anticipated cost rate (ETC). To do this, we must first adjust the values of the other parameters and costs are $C_1 = 7, C_2 = 8, C_3 = 6, C_4 = 3, C_5 = 4,$ $C_6 = 5, p_{10} = 0.6, p_{20} = 0.5, p_{12} = 0.4, p_{21} = 0.5$. For different values of β_1 and β_2, μ_1 and μ_2, λ_1 and λ_2, the graphs of the total expected cost are shown in Figs. 2–11. A numerical study is provided in the following figures to demonstrate the system's sensitivity to the effect of changing various parameters. From Figs. 2–11, we observer the following:

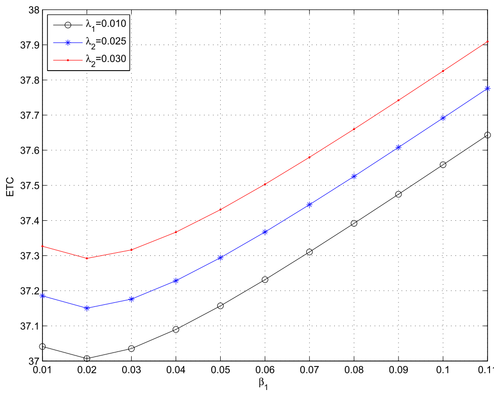

Fig. 2. Effect of λ_1 for different values of β_1 on expected total cost (ETC): Fix values, $\mu_1 = 3.0, \mu_2 = 2.0, \lambda_2 = 0.03, \beta_2 = 0.08, \theta_1 = \theta_2 = 2$.

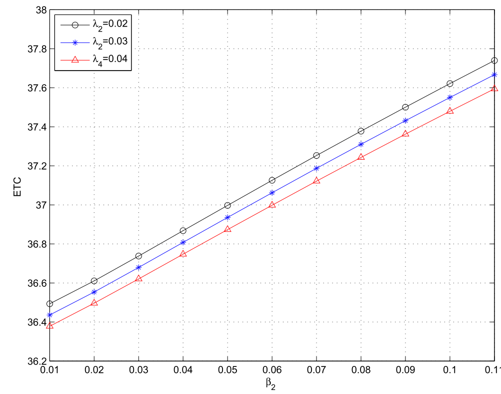

Fig. 5. Effect of λ_2 for different values of β_2 on expected total cost (ETC): Fix values, $\mu_1 = 3.0, \mu_2 = 2.0, \lambda_1 = 0.02, \beta_1 = 0.07, \theta_1 = \theta_2 = 2$.

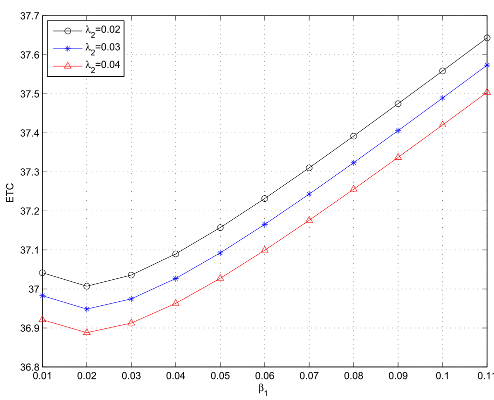

Fig. 3. Effect of λ_2 for different values of β_1 on expected total cost (ETC): fix values, $\mu_1 = 3.0, \mu_2 = 2.0, \beta_1 = 0.03, \beta_2 = 0.08, \theta_1 = \theta_2 = 2$.

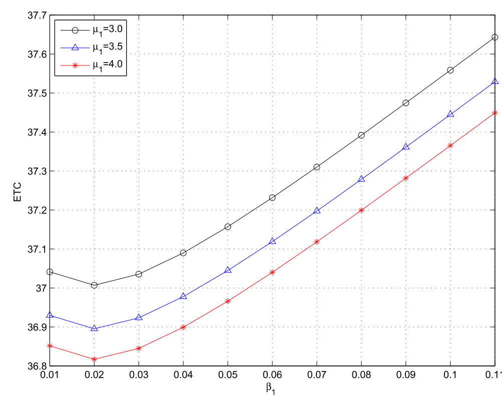

Fig. 6. Effect of μ_1 for different values of β_1 on expected total cost (ETC): Fix values, $\mu_2 = 2.0, \lambda_1 = 0.02, \lambda_2 = 0.03, \beta_2 = 0.08, \theta_1 = \theta_2 = 2$.

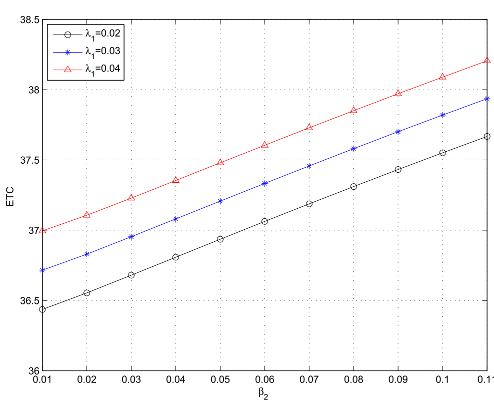

Fig. 4. Effect of λ_1 for different values of β_2 on expected total cost (ETC): Fix values, $\mu_1 = 2.0, \mu_2 = 3.0, \lambda_2 = 0.03, \beta_1 = 0.07, \theta_1 = \theta_2 = 2$.

(i) Fig. 2 depicts the estimated total cost for various values of λ_1 throughout the provided range of β_1. It is clear from the graph that increasing the arrival rates increases the anticipated total cost (ETC) by a substantial amount. It has a great impact on the holding cost of the customers in the system. The global minimum value of ETC occurs at the value of $\lambda_1 = 0.010$ and $\beta_1 = 0.02$. Hence ETC is more sensitive on β_1 to compare λ_1. Similarly, over the given range of β_1, with respect to any value of λ_2 from the Fig. 3, it has been noted that if the arrival rates are raised, the anticipated total cost (ETC) will be reduced significantly. The global minimum value of ETC occurs at value of $\lambda_2 = 0.04$ and $\beta_1 = 0.02$. Hence arrival rates are vital to the system corresponding to the values of perishable rates. so it is indicates ETC is more sensitive on β_1 to compare λ_1 and λ_2 to the system.

(ii) From the Fig. 4, Over the given range of β_2, concerning any value of λ_1, it has been shown that increasing arrival rates increases the estimated total cost (ETC) by a substantial amount. The arrival rates increase indicates cost increases as because of less cost of customers lost and less waiting time to the system. The minimum value of ETC occurs at the lowest value of λ_1 corresponding to the lowest value of β_2. Similarly, over the given range of β_2, with respect to any value of λ_2 from the Fig. 5, it can be observed that increasing the arrival rates reduces the expected total cost (ETC) by a substantial amount. The minimum value of ETC occurs at the lowest value of λ_2 corresponding to the lowest value of β_2. From the observation it seems that ETC is more sensitive on β_2 to compare λ_1 and λ_2 to the system.

(iii) From the Fig. 6, it is clear that if the service rates are increased, the estimated total cost(ETC) is decreased by a considerable amount, over the given range of β_1, corresponding to each value of μ_1. The global minimum value of ETC occurs at the values of $\mu_1 = 4.0$ and $\beta_1 = 0.02$. From the observation it seems that service rates are very sensitive to the system. Similarly, we see that from the Fig. 7, the optimal expected cost rate also increases when β_1 increases for various values of μ_2 increase. The global minimum value of ETC occurs at the values of $\mu_2 = 2.0$ and $\beta_1 = 0.02$. Hence ETC is more sensitive on β_1 to compare μ_1 and μ_2.

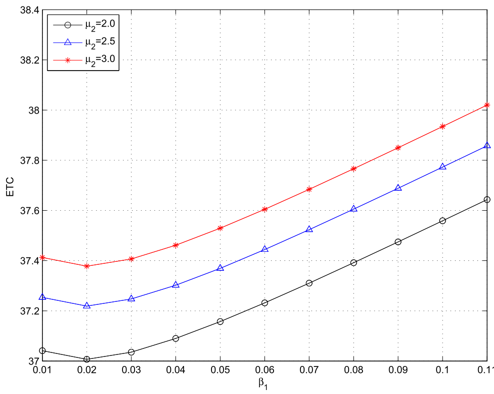

Fig. 7. Effect of μ_2 for different values of β_1 on expected total cost (ETC): Fix values, $\mu_1 = 3.0, \lambda_1 = 0.02, \lambda_2 = 0.03, \beta_2 = 0.08, \theta_1 = \theta_2 = 2$.

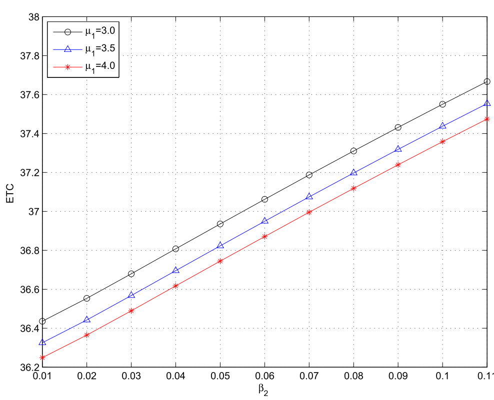

Fig. 8. Effect of μ_1 for different values of β_2 on expected total cost (ETC): Fix values, $\mu_2 = 2.0, \lambda_1 = 0.02, \lambda_2 = 0.03, \beta_1 = 0.07, \theta_1 = \theta_2 = 2$.

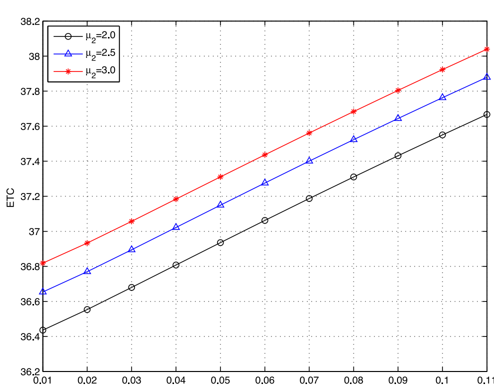

Fig. 9. Effect of μ_2 for different values of β_2 on expected total cost (ETC): Fix values, $\mu_1 = 3.0, \lambda_1 = 0.02, \lambda_2 = 0.03, \beta_1 = 0.07, \theta_1 = \theta_2 = 2$.

(iv) As seen in Fig. 8, increasing the service rates decreases the estimated total cost significantly, over the given range of β_2, corresponding to each value of μ_1. The minimum value of ETC occurs when μ_1 is largest and β_2 is lowest. From the observation it seems that service rates are very sensitive to the system. Similarly, we see that from the Fig. 9, the optimal expected cost rate also increases when β_2 rises for various values of μ_2. The minimum value of ETC occurs at the lowest value of both μ_2 and β_2. Hence ETC is more sensitive on β_2 to compare μ_1 and μ_2.

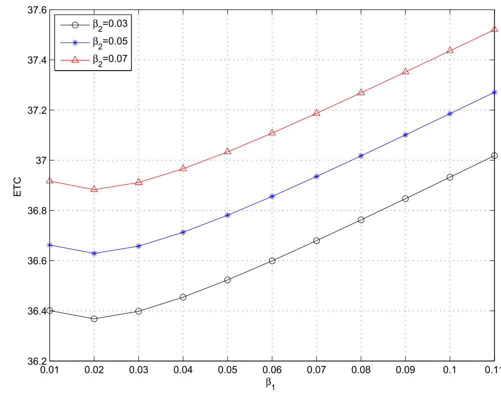

Fig. 10. Effect of β_2 for different values of β_1 on expected total cost (ETC): Fix values, $\mu_1 = 2.0, \mu_2 = 3.0, \lambda_1 = 0.02, \lambda_2 = 0.03, \theta_1 = \theta_2 = 2$.

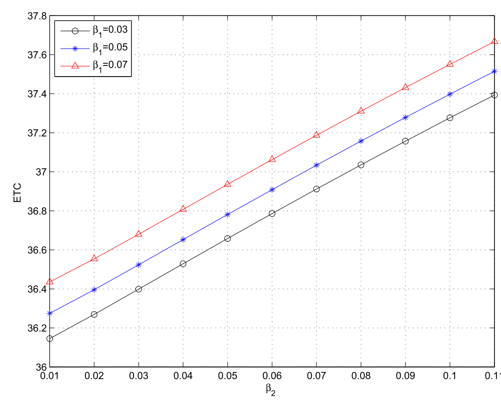

Fig. 11. Effect of β_1 for different values of β_2 on expected total cost (ETC): Fix values, $\mu_1 = 2.0, \mu_2 = 3.0, \lambda_1 = 0.02, \lambda_2 = 0.03, \theta_1 = \theta_2 = 2$.

(v) As a result of this observation, we can see in Fig. 10 that raising the perishable rates significantly increases the estimated overall cost, over the given range of β_1, corresponding to each value of β_2. The global minimum value of ETC occurs at the values of $\beta_2 = 0.03$ and $\beta_1 = 0.02$. From the observation it seems that perishable rates are very sensitive to the system. Similarly, we see that from the Fig. 11, the optimal expected cost rate also increases when β_2 rises for various values of β_1. The minimum value of ETC occurs at the lowest values of both β_1 and β_2. Hence ETC is more sensitive on β_1 to compare β_1 and β_2.

10. Conclusion

In this study, we built a Jackson network stochastic queuing model with a (s, S) policy for level dependent perishable inventory. Matrix Analytic method have been applied to mathematical analysis, modeling, optimization of queuing inventory system of the model. Several parameters are explored based on measures of system performance. Due to the level dependent perishable items in the model, a sensitivity analysis of the anticipated total cost function with respect to various parameters is graphically performed. The proposed stochastic model has become integral part of organizational decision making such as capacity planing, facility planning, scheduling, managing perishable or non-perishable inventories, quality improvement of the organization's products or services and more. In this paper the obtained results of level dependent perishable inventory system of Jackson network model has potential application in engineering of communication networks, production-inventory systems, traffic and transportation systems as the model of practical applications. The main purpose of the optimization is to minimize the total expected cost of the systems. For future study, the model can be developed in several directions such as, (i) considering reneging and jockeying customers and (ii) in the present work considered only the Poisson process, but it is possible to incorporate different demand processes, such as phase-type distribution and the Markovian arrival process (MAP).

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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