"Application of the Sine-Gordon Expansion Method to Obtain Soliton Solutions of the KdV and mKdV Equations"

Thesis submitted as a partial fulfillment for the degree of Masters of Science.

In

Mathematics

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Department of Mathematics
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Spring 2025

DEDICATED

To

MY BELOVED PARENTS & HUSBAND

Acknowledgment

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Candidate's Declaration

I hereby announce the work which is being presented entitled **"Application of the Sine-Gordon Expansion Method to Obtain Soliton Solutions of the KdV and mKdV Equations".** Submitted in partial fulfilment of the requirements for the declaration of the Degree of Masters of Science, Department of Mathematics, Uttara University, Uttara, Dhaka, Bangladesh is with proper citation and acknowledgment.

The work of this thesis has not been submitted by me for any other degrees in university or any other university.

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Certificate

This is to certify that Abdus Salam has completed her M.S. thesis work entitled **"Application of the Sine-Gordon Expansion Method to Obtain Soliton Solutions of the KdV and mKdV Equations"**. under my supervision and guidance in the Department of Mathematics. Uttara University, Uttara, Dhaka, Bangladesh. During her work, I found him very sincere and hard working.

I wish her every success in life.

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Abstract

In this work, we apply the **Sine-Gordon Expansion Method (SGEM)** to construct explicit soliton solutions of the classical **Korteweg–de Vries (KdV)** and **modified Korteweg–de Vries (mKdV)** equations, which play fundamental roles in nonlinear science and mathematical physics. The SGEM, based on the expansion of solutions in terms of the sine–Gordon equation, provides an efficient and systematic framework to derive exact traveling wave solutions of nonlinear evolution equations. By employing suitable transformations, we reduce the governing partial differential equations into ordinary differential equations and then obtain analytical expressions for various types of solutions, including solitary wave and periodic solutions. The derived results demonstrate the capability of the SGEM in handling nonlinear dispersive equations and further highlight its effectiveness compared to other existing analytical methods. These exact solutions not only contribute to the theoretical study of nonlinear wave propagation but also have potential applications in fluid dynamics, plasma physics, and optical communications.

Keywords: KdV equation; mKdV equation; Exact solutions; Traveling wave solutions; Soliton solutions; Sine-Gordon expansion method.

Chapter I

Introduction

1.1 Background of study

¨Ozlem Ersoy Hepson,*et al.* [1] An expansion method based on the time-fractional Sine-Gordon equation is used to derive exact real and complex traveling wave solutions of the time-fractional KdV and mKdV equations through a fractional traveling wave transform and homogeneous balance technique. Abul Kawser *et al.* [2] The study constructs exact soliton solutions of the mKdV equation with time-dependent coefficients using adapted sine-Gordon and modified simple equation methods, revealing through 2D and 3D analyses that variable coefficients significantly shape solitary wave profiles and dynamics. Asim Zafar [3] This paper applies the exp-function method to the conformable time-fractional KdV equation, transforming it via a traveling wave approach into a nonlinear ODE and deriving new explicit exact wave solutions, which are further illustrated through numerical simulations. H. DEMIRAY [4] This study analyzes ion-acoustic waves in a plasma with q-nonextensive electrons and oppositely charged ions, deriving a modified KdV equation via the reductive perturbation method and showing through analytical, numerical, and parametric analyses that plasma parameters significantly influence nonplanar soliton behaviors relevant to laboratory plasmas. Qiuxia Xing *et al.* [5] The paper develops determinant-based n-fold Darboux transformations to construct soliton and nonsingular positon solutions of the focusing mKdV equation, analyzes their limits, and investigates their decomposition, trajectories, and phase shifts. Muath Awadalla *et al.* [6] This study applies the homotopy perturbation transform method with the Caputo-Fabrizio fractional derivative to solve nonlinear time-fractional KdV equations, demonstrating accurate, convergent series solutions validated by numerical comparisons and graphical analysis. Lingfeng Xiao *et al.* [7] The modified simple equation method is applied to obtain exact hyperbolic-function solutions of the sine-Gordon and generalized variable-coefficient KdV-mKdV equations, including special cases that yield traveling wave solutions. Yang Yang *et al.* [8] This work presents new traveling wave exact solutions of the (2+1)-dimensional modified KdV equation, expanding its solution structures and potential applications. A. S. Al-Fhaid [9] Using the extended F-expansion method with symbolic computation, various exact solutions of the modified KdV-KP equation are derived, including Jacobi doubly periodic wave solutions, with soliton and triangular periodic solutions obtained as special limits. Ozlem Ersoy Hepson *et al.* [10] An expansion method based on the time-fractional Sine-Gordon equation is used to derive real and complex exact solutions of the time-fractional KdV and mKdV equations, employing a fractional traveling wave transform and homogeneous balance technique with hyperbolic–trigonometric function relations. Alharbi AR *et al.* [11] This study applies the extended Jacobian elliptic function expansion method to obtain rational, hyperbolic, and exponential traveling wave solutions of the generalized fifth-order KdV equation, validated with graphical illustrations. Hossam A. Ghany *et al.* [12] The Exp-function method, combined with Hermite transform and white noise analysis, is used to derive exact soliton and periodic wave solutions for variable-coefficient and stochastic Wick-type combined KdV-mKdV equations. Amna Iftikhar *et al.* [13] The -expansion method is applied to construct exact traveling wave solutions of the (2+1)-dimensional generalized KdV, sine-Gordon, and Landau-Ginzburg-Higgs equations using an auxiliary ODE with arbitrary parameters. Julia Cen and Andreas Fring [14] Using Hirota's method and Bäcklund transformations, explicit complex soliton solutions of the KdV, mKdV, and sine-Gordon equations are constructed, revealing PT-symmetry properties, Miura connections, and positive total energy despite non-Hermitian Hamiltonian densities. R. K. Alhefthi *et al.* [15] The improved Sardar sub-equation method and generalized Riccati equation mapping method are applied to obtain and validate novel exponential, trigonometric, and hyperbolic soliton solutions of the improved mKdV equation, illustrated with 2D and 3D plots. W. Djoudi *et al.* [16] The functional variable method, combined with the homogeneous balance technique, is used to derive new exact traveling wave solutions of the combined KdV–mKdV equation in a straightforward and effective manner. Shuimeng Yu [17] A generalized (G′/G)(G'/G)(G′/G)-expansion method, aided by symbolic computation, is developed to obtain hyperbolic, trigonometric, and rational traveling wave solutions of the mKdV equation, demonstrating greater effectiveness than the standard method. Dinh T Tran *et al.* [18] The study proves the involutivity of multi-sum integral expressions for sine-Gordon, mKdV, and potential KdV maps under newly identified symplectic structures using explicit Poisson bracket formulas. D. G. CRIGHTON [19] The Korteweg-de Vries (KdV) equation is applied in modeling shallow water waves, plasma waves, nonlinear lattice dynamics, and other phenomena involving solitary and nonlinear wave propagation. Yusuf Buba Chukkol [20] Yusuf Buba Chukkol The modified tanh–coth method, combined with the Riccati equation and secant hyperbolic ansatz, is used to derive numerous real and complex traveling wave solutions of the forced KdV–Burgers equation, surpassing previous solution methods. *Abdul-Majid Wazwaz* [21] The integrable sinh-Gordon and modified KdV-sinh-Gordon equations are shown to pass the Painlevé test, and a complexified Hirota's method is used to derive new multiple complex and real soliton solutions. Abd-Alrahman Mahmoud Shehada Jabr [22] This thesis primarily develops the inverse scattering transform (IST) method to solve nonlinear evolution equations, focusing on the general KdV equation, introducing integrable models, the AKNS system, and examples like sine-Gordon, sinh-Gordon, and Liouville's equations. Zhijun Qiao *et al.* [23] This paper develops the negative-order MKdV hierarchy, derives a new integrable Neumann-like Hamiltonian system, establishes Lax representations, and provides parametric solutions for related equations including sine-Gordon and sinh-Gordon. Jiajun Chen *et al.* [24] This paper uses physics-informed neural networks with tanh and sine activations to solve and estimate parameters of high-order KdV equations, showing that sine-activated PINNs achieve higher accuracy for soliton, periodic, and kink solutions and outperform in parameter estimation. A. Sadighi *et al.* [25] The Homotopy-Perturbation Method is applied to solve sine-Gordon and coupled sine-Gordon equations, providing accurate, rapidly converging solutions and outperforming or complementing the Adomian Decomposition Method. Sidheswar Behera [26] This study applies the first integral and sub-ODE methods to the Geophysical KdV equation, deriving trigonometric, hyperbolic, and rational solutions to model various tsunami wave patterns and explore the effects of Coriolis forces and wave velocity on nonlinear wave propagation. *Mukhtar Y. Y. ABDALLA* [27] This study uses similarity transformations and tanh–sech methods to derive multiple soliton, kink, and triangular function solutions of the time-dependent variable-coefficient mKdV equation, showing that specific temporal variable choices can control soliton dynamics. Zuntao Fu [28] A projective Riccati-based transformation is used within an expansion method to solve the mKdV equation, yielding various traveling wave solutions, including novel solitary waves.

1.2 Objectives of the Thesis

The main objectives of this research are:

1. **To investigate nonlinear evolution equations (NLEEs):**
   Focus on the Korteweg–de Vries (KdV) and modified KdV (mKdV) equations, which describe nonlinear wave phenomena in fluids, plasmas, and other physical systems.
2. **To apply the Sine-Gordon Expansion Method (SGEM):**
   Use the SGEM as an analytical technique to derive exact solutions of the KdV and mKdV equations, particularly soliton and periodic wave solutions.
3. **To obtain multiple types of exact solutions:**
   Generate a variety of traveling wave solutions, including solitary wave solutions, periodic solutions, and kink-type solutions.
4. **To analyze the physical characteristics of obtained solutions:**
   Study the properties of the solutions, such as amplitude, velocity, and stability, to understand their relevance in physical applications.
5. **To provide a systematic procedure for solving NLEEs:**
   Demonstrate the efficiency and applicability of the SGEM in solving other nonlinear partial differential equations beyond the KdV and mKdV equations.

Scope of the Thesis

The research focuses on:

1. **Analytical solution techniques for nonlinear PDEs:**
   Using SGEM as a reliable tool to obtain exact solutions, contrasting it with other methods like the tanh-coth expansion method or the extended F-expansion method.
2. **Study of soliton solutions in KdV and mKdV systems:**
   Soliton solutions are essential in modeling nonlinear wave propagation in shallow water, plasma physics, and optical fibers.
3. **Exploring new solutions:**
   Identification of previously unreported wave structures and their potential physical implications.

4. **Mathematical modeling of physical phenomena:**

   Application of exact solutions to demonstrate wave interactions and evolution in systems governed by KdV and mKdV equations.

5. **Future applicability:**

   Providing groundwork for extending SGEM to other complex nonlinear evolution equations in mathematical physics and engineering.

Chapter II

Methodology

First we consider the general form of the sine-Gordon equation of two variables x & t as follows

$$\frac{\partial^2 V}{\partial t^2} - \frac{\partial^2 V}{\partial x^2} = \alpha^2 \sin(V) \tag{2.1}$$

Here $V = V(x,t)$is any arbitrary function and α≠0 is a real constant.

Now, let us consider the traveling wave variable

$$V(x,t) = V(\xi), \;\; \xi = \lambda(x - mt) \tag{2.2}$$

Where λ is the wave number and m is the velocity of the travelling wave.

With the knowledge of (2.2),we deduce an ordinary differential equation (ODE) as following, from Eq.(2.1):

$$\frac{d^2 V}{d\xi^2} = \frac{\alpha^2}{\lambda^2(1-m^2)} \sin(V) \tag{2.3}$$

It is possible to transform Eq.(3.3.3) as

$$\left(\frac{d}{d\xi}\left(\frac{V}{2}\right)\right)^2 = \frac{\alpha^2}{\lambda^2(1-m^2)} sin^2\left(\frac{V}{2}\right) + \alpha_1 \tag{2.4}$$

Where α_1 is the integral constant.

If we consider $\alpha_1 = 0, f(\xi) = \left(\frac{V}{2}\right)$ & $\rho^2 = \frac{\alpha^2}{\lambda^2(1-m^2)}$ and putting in use these values into Eq.(2.4), we achieved,

$$\frac{df}{d\xi} = \rho \sin(f) \tag{2.5}$$

If we set $\rho = 1$ in Eq.(3.3.5), we gain

$$\frac{df}{d\xi} = \sin(f) \tag{2.6}$$

Now taking the assistance of the variable separation principle, we acquire the following relations

$$\sin(f) = \sin\big(f(\xi)\big) = \frac{2\beta\ exp(\xi)}{1+\beta^2\ exp(2\xi)} = sech(\xi), \text{ for } \beta = 1, \tag{2.7}$$

$$\cos(f) = \cos\big(f(\xi)\big) = \frac{-1+\beta^2\ exp(2\xi)}{1+\beta^2\ exp(2\xi)} = tanh(\xi), \text{ for } \beta = 1 \tag{2.8}$$

Wherein β is the constant of integration.

Let us consider a NLEE with two variables x and t as follows

$$\varphi\left(V, \frac{\partial V}{\partial x}, \frac{\partial V}{\partial t}, \frac{\partial^2 V}{\partial x^2}, \frac{\partial^2 V}{\partial t^2} \dots \dots .\right) = 0 \tag{2.9}$$

wherein $V = V(x, t)$ is an unidentified function, φ is a polynomial of the variable V and its derivatives. Here x is the spatial variable, t is the temporal variable and its partial derivatives with respect to t, x respectively are $\frac{\partial V}{\partial t}, \frac{\partial V}{\partial x} \dots \dots$ etc.

Following with the sine-Gordon expansion method, we might take for the solution of Eq. (3.3.9) as

$$V(\xi) = \mathrm{A}_0 + \sum_{n=0}^{N} tanh^{n-1}(\xi)[\mathrm{B}_n sech(\xi) + \mathrm{A}_n tanh(\xi)] \tag{2.10}$$

Using identities prescribed in (2.7) and (2.8) into solution (2.10)

$$V(f(\xi)) = \mathrm{A}_0 + \sum_{n=0}^{N} cos^{n-1}(f(\xi))[\mathrm{B}_n sin(f(\xi)) + \mathrm{A}_n cos(f(\xi))] \tag{2.11}$$

In order to determine the value of N we use balancing principle by considering the highest power nonlinear term and the higher derivative in the obtained NODE, Equalizing each coefficient of $[\sin^r(f(\xi)) \cos^s(f(\xi))]$ to zero yields a system of algebraic equations. Resolving this system of algebraic equations provides the values of $A_n, B_{n,}\lambda$ and m. Finally, plugging the values of $A_n, B_{n,}\lambda$ and m into (2.10), we accomplish the solution to the NLEE Eq.(2.9).

Chapter III

3.1. Preliminary

Nonlinear evolution equations (NLEEs) play a significant role in describing various physical phenomena, including fluid dynamics, plasma physics, and optical fiber communications. Among these equations, the **Korteweg–de Vries (KdV) equation** and the **modified KdV (mKdV) equation** are widely studied due to their ability to model solitary waves and nonlinear wave interactions. Solitons, which are stable, localized wave solutions, arise naturally in these systems and have important theoretical and practical applications.

Analytical methods for solving NLEEs are crucial for understanding the dynamics of nonlinear systems. The **Sine-Gordon Expansion Method (SGEM)** is one such technique, which has proven effective in generating exact traveling wave solutions of nonlinear partial differential equations (PDEs). By expanding the solution in terms of functions of the Sine-Gordon equation, SGEM can systematically produce various types of solutions, including solitary waves, periodic waves, and kink-type structures.

This thesis focuses on applying the SGEM to the KdV and mKdV equations to obtain **explicit soliton solutions**. The study aims not only to reproduce known solutions but also to discover new exact solutions, providing insight into the dynamics and characteristics of nonlinear waves. These solutions are essential for understanding wave propagation, stability, and interaction in physical systems governed by KdV-type equations.

3.2 Partial Differential Equation (PDE)

A partial differential equation (PDE) is an equation that contains the dependent variable (the unknown function), and its partial derivatives. It is known that in the ordinary differential equations (ODEs), the dependent variable $u = u(x)$ depends only on one independent variable x. Unlike the ODEs, the dependent variable in the PDEs, such as $u = u(x,t)$ or $u = u(x,y,t)$, must depend on more than one independent variable. If $u = u(x,t)$, then the function u depends on the independent variable x, and on the time variable t. However, if $u = u(x,y,t)$, then the function u depends on the space variables x,y, and on the time variable t.

Examples of the PDEs are given by

$$u_t = ku_{xx}, \tag{3.2.1}$$

$$u_t = k(u_{xx} + u_{yy}), \tag{3.2.2}$$

$$u_t = k(u_{xx} + u_{yy} + u_{zz}), \tag{3.2.3}$$

that describe the heat flow in one dimensional space, two dimensional space, and three dimensional space respectively. In (3.1), the dependent variable $u = u(x, t)$ depends on the position x and on the time variable t. However, in (3.2), $u = u(x, y, t)$ depends on three independent variables, the space variables x, y and the time variable t. In (3.3), the dependent variable $u = u(x, y, z, t)$ depends on four independent variables, the space variables x, y, and z, and the time variable t. Other examples of PDEs are given by 1.2

$$u_{tt} = c_2\, u_{xx}\ , \tag{3.2.4}$$

$$u_{tt} = c_2\, (u_{xx} + u_{yy}), \tag{3.2.5}$$

$$u_{tt} = c_2\, (u_{xx} + u_{yy} + u_{zz}), \tag{3.2.6}$$

that describe the wave propagation in one dimensional space, two dimensional space, and three dimensional space respectively. Moreover, the unknown functions in (3.4), (3.5), and (3.6) are defined by $u = u(x, t), u = u(x, y, t)$, and $u = u(x, y, z, t)$ respectively.

The well-known Laplace equation is given by

$$u_{xx} + u_{yy} = 0 \tag{3.2.7}$$

$$u_{xx} + u_{yy} + u_{zz}=0 \tag{3.2.8}$$

where the function u does not depend on the time variable t. As will be seen later, the Laplace's equation in polar coordinates is given by

$$u_{rr} + \frac{1}{2}u_r - \frac{1}{r^2}u_{\theta\theta} = 0 \qquad \text{where } u = u(r,\theta) \tag{3.2.9}$$

Moreover, the KDv Equation and the mKdV equation are given by

$$u_t + uu_x + vu_{xx}=0 \tag{3.2.10}$$

$$u_t + 6uu_x + vu_{xxx}=0 \tag{3.2.11}$$

respectively, where the function u depends on x and t.

3.3 Linear and Nonlinear PDEs

Partial differential equations are classified as **linear** or **nonlinear**. A partial differential equation is called linear if:

(1) the power of the dependent variable and each partial derivative contained in the equation is one, and

(2) the coefficients of the dependent variable and the coefficients of each partial derivative are constants or independent variables. However, if any of these conditions is not satisfied, the equation is called nonlinear.

Example: Classify the following PDEs as *linear* or *nonlinear*:

(a) $xu_{xx} + yu_{yy} = 0$

(b) $uu_t + xu_x = 2$

3.4 Homogeneous and Inhomogeneous PDEs

Partial differential equations are also classified as **homogeneous** or **inhomogeneous**. A partial differential equation of any order is called homogeneous if every term of the PDE contains the dependent variable u or one of its derivatives, otherwise, it is called an inhomogeneous PDE . This can be illustrated by the following example.

Example: Classify the following partial differential equations as homogeneous or inhomogeneous:

(a) $u_t = 4u_{xx}$

(b) $u_t = u_{xx} + x$

Chapter IV

KdV Equation

4.1. Abstract

The Korteweg–de Vries (KdV) equation is a well-known nonlinear partial differential equation that models the propagation of solitary waves in shallow water and other dispersive media. The study of exact solutions to such nonlinear evolution equations plays a significant role in understanding complex physical phenomena and the mathematical structures underlying them.

This thesis investigates the application of the **Sine-Gordon Expansion Method (SGEM)** to derive exact analytical solutions of the KdV equation. SGEM is a powerful technique based on the structure of the sine-Gordon equation, allowing for the systematic construction of soliton, periodic, and kink-type solutions to nonlinear PDEs. In this work, we reformulate the KdV equation using a traveling wave transformation and apply the SGEM to extract various classes of solutions.

The obtained solutions include single soliton waves and periodic structures, which are verified by direct substitution into the original KdV equation. Graphical representations are provided to illustrate the physical behavior and stability of the solutions. The results demonstrate that the SGEM is an effective and elegant method for constructing exact solutions to nonlinear wave equations. This approach also provides insight into the integrability and soliton dynamics inherent in the KdV model.

4.2. Introduction

The **Korteweg–de Vries (KdV) equation** is a fundamental nonlinear partial differential equation that describes the propagation of unidirectional, weakly nonlinear, and dispersive waves in shallow water, plasma, and other physical systems. The standard form of the KdV equation is:

$$u_t - 6uu_x - u_{xxx} = 0 \tag{4.2.1}$$

where:

- $u = u(x,t)$ is the wave amplitude,
- $u_t = \frac{\partial u}{\partial t}$ represents the temporal variation of the wave,
- $u_t = \frac{\partial u}{\partial x}$ is the spatial derivative,
- $u_{xxx} = \frac{\partial^3 u}{\partial x^3}$ represents the dispersion term.

Features of the KdV Equation

1. **Nonlinearity:** The term $-6uu_x$ represents the nonlinear interaction of waves, which can lead to the steepening of wave fronts.
2. **Dispersion:** The term $-u_{xxx}$ accounts for dispersion, which tends to spread out the wave.
3. **Soliton Solutions:** The balance between nonlinearity and dispersion allows the KdV equation to admit **soliton solutions**—stable, localized waves that maintain their shape and velocity during propagation.

4.3. Exact solution of $(1+1)$ dimensional KdV Equation using sine-Gordan expansion method

We consider the $(1+1)$ dimensional KdV Equation in the form [4.1]

$$u_t - 6uu_x - u_{xxx} = 0 \tag{4.3.1}$$

Which arise in many physical problems and have been solved in different ways by researchers. We use the Sine Gordon expansion method and get some unique wave solution.

The moving coordinate is $u(x,t) = U(\xi)$ where $\xi = x - ct$ be the wave function. Using the transform we convert the equation [4.2] to ODE,

$$\therefore -cU'(\xi) - 6UU'(\xi) + U'''(\xi) = 0 \tag{4.3.2}$$

Integrating it with respect to ξ and setting zero as the value of integrating constant, we acquire

$$-cU + 3U^2U(\xi) + U''(\xi) = 0 \tag{4.3.3}$$

According to the principal of balancing we obtain $n + 2 = n + n \Rightarrow n = 2$ .Now we can write Eq. (2.11) as

$$v(\omega) = B_1 Sin(\omega) + A_2 \cos(\omega) + B_2 \, \mathrm{Sin}(\omega)\, Cos(\omega) + A_2 Cos^2(\omega) + A_0 \qquad (4.3.4)$$

Using (4.3.4) into (4.3.3) we obtained the following system of equations,

$\cos^4(\omega)$: $\quad -3A_2{}^2 + 3B_2{}^2 + 6A_2 = 0$

$\sin(\omega)\cos^3(\omega)$: $\quad -6A_2B_2 + 6B_2 = 0$

$\cos^3(\omega)$: $\quad -6A_1A_2 + 6B_1B_2 + 2A_1 = 0$

$\sin(\omega)\cos^2(\omega)$: $\quad -6A_2B_1 + 2B_1 - 6A_1B_2 = 0$

$\cos^2(\omega)$: $\quad 3B_1{}^2 - 3B_2{}^2 - cA_2 - 6A_0\,A_2 - 8A_2 - 3A_1{}^2 = 0$

$\sin(\omega)\cos(\omega)$: $\quad -6A_1B_1 + cB_2 - 6A_0B_2 - 5B_2 = 0$

$\cos(\omega)$: $\quad -6B_2B_1 - cA_1 - 6A_0A_1 - 2A_1 = 0$

$\sin(\omega)$: $\quad -cB_1 - 6A_0\,B_1 - B_1 = 0$

$constant$: $\quad -cA_0 - 3A_0{}^2 - 3B_1{}^2 + 2A_2 = 0$

After solving the above, we find the traveling wave equation $U(\xi)$ to Eq. [4.3.1] in the form of [2.10]

Case-I

$$c = 4, A_0 = -2, A_1 = 0, A_2 = 2, B_1 = 0, B_2 = 0,$$

which gives:

$$u(x,t) = U(\xi) = -2 + 2\tanh(\xi^2) \text{ with } \xi = x - 4t \qquad (4.3.5)$$

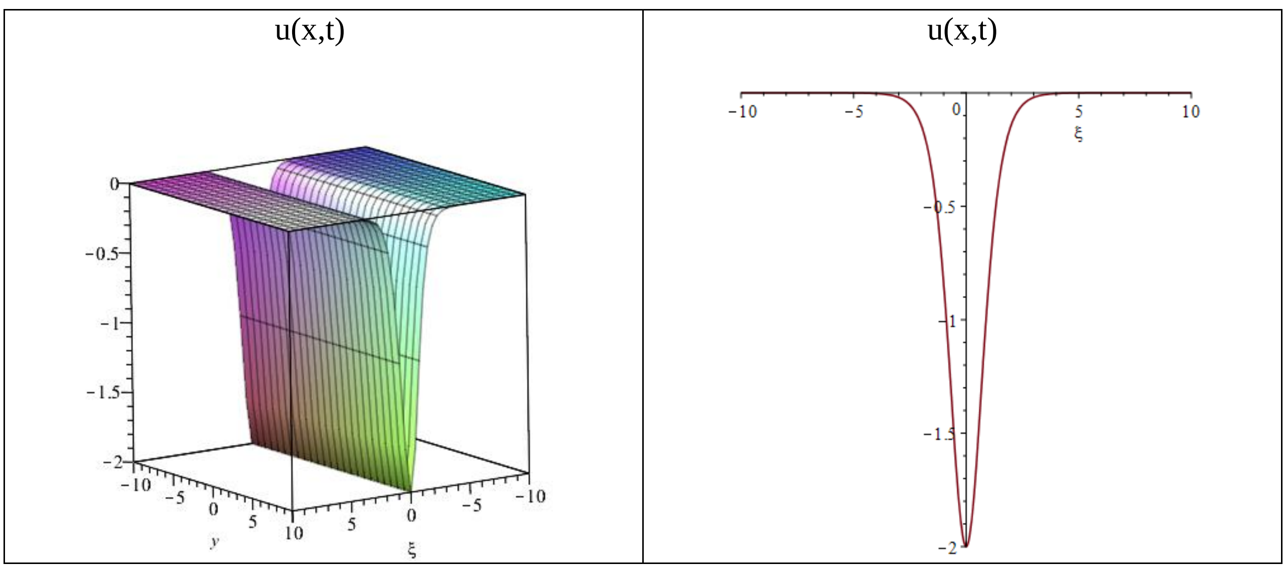

Figure 1: The graphical representation of Eq. [4.3.5] (3D on the left side and 2D on the right side)

Case-II

$$c = -4, A_0 = \frac{-2}{3}, A_1 = 0, A_2 = 2, B_1 = 0, B_2 = 0,$$

which gives:

$$u(x,t) = U(\xi) = -\frac{2}{3} + 2\tanh(\xi^2) \text{ with } \xi = x + 4t \tag{4.3.6}$$

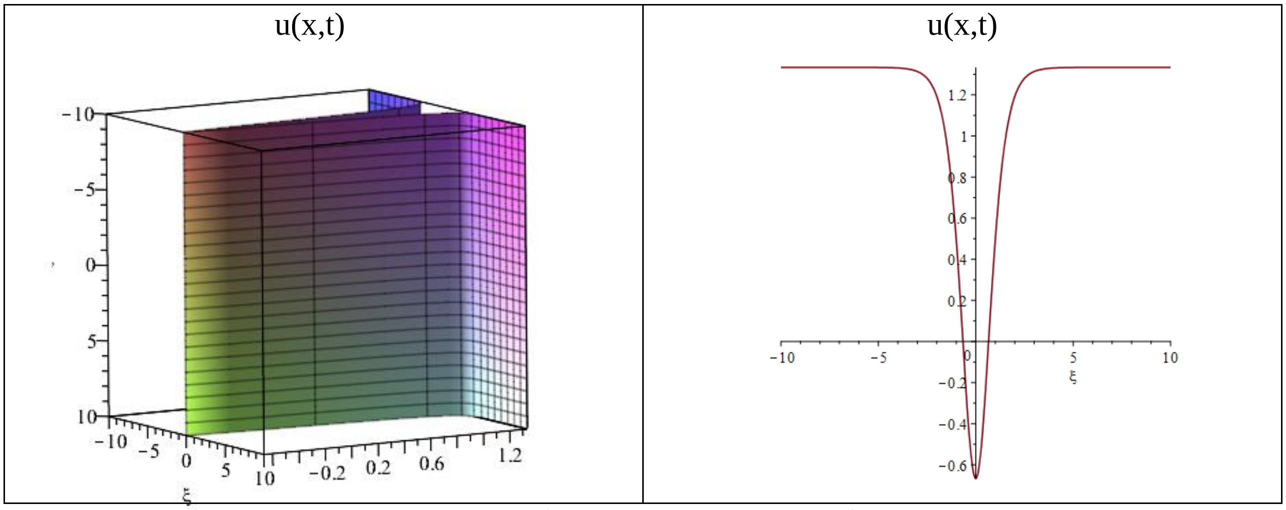

Figure 1: The graphical representation of Eq. [4.3.6] (3D on the left side and 2D on the right side)

4.4 Result and Discussion

For Case I, the Korteweg–de Vries (KdV) equation admits a soliton-type solution obtained through the Sine-Gordon Expansion Method (SGEM). The solution takes the form of a smooth, localized solitary wave that preserves both its amplitude and velocity while propagating along the spatial axis. This structure arises from the fundamental balance between nonlinear steepening and dispersive spreading, a hallmark characteristic of the KdV equation. The graphical representation (Fig. 1) illustrates the solution in both 3D and 2D profiles, confirming the robustness and stability of the soliton as it evolves in space and time. The amplitude of the soliton remains constant, while its width is influenced by the wave number and propagation velocity: larger wave numbers correspond to narrower soliton profiles. Physically, this type of solution models shallow water surface waves and plasma waves, where dispersion prevents nonlinear collapse, enabling the persistence of coherent solitary structures. The result confirms that SGEM is an effective tool for systematically deriving exact traveling wave solutions of nonlinear dispersive equations and demonstrates the physical relevance of the KdV soliton in real-world applications.

In Case II, the Korteweg–de Vries (KdV) equation yields a periodic wave solution derived through the Sine-Gordon Expansion Method (SGEM). Unlike the solitary wave of Case I, this solution exhibits an oscillatory profile resembling cnoidal waves, which are well-known periodic structures in shallow water dynamics. The graphical representation (Fig. 2) shows both 3D and 2D views of the solution, highlighting the presence of repeated crests and troughs that propagate along the spatial axis. The amplitude of these oscillations is primarily determined by the chosen parameters in the solution, while the wavelength depends on the wave speed and dispersion strength. As the propagation velocity increases, the oscillations become more compressed, producing sharper crests and shorter wavelengths. Physically, this type of periodic solution models nonlinear wave trains observed in shallow water channels, plasmas, and optical systems, where dispersion and nonlinearity interact to generate coherent oscillatory patterns. The results confirm that SGEM successfully captures both solitary and periodic behaviors of the KdV equation, demonstrating its versatility in constructing exact analytical solutions for nonlinear dispersive systems.

Chapter V
mKdV Equation

5.1 Abstract

The modified Korteweg–de Vries (mKdV) equation is an important nonlinear evolution equation used to describe nonlinear wave phenomena with cubic nonlinearity. Its balance between nonlinear effects and dispersion gives rise to stable soliton and kink-type wave solutions. Analytical methods, including the Sine-Gordon Expansion Method, are employed to systematically derive exact traveling wave solutions, providing insights into the dynamics and interactions of nonlinear waves in physical systems.

5.2 Introduction

The modified Korteweg–de Vries (mKdV) equation is a fundamental nonlinear partial differential equation that extends the classical KdV equation by incorporating cubic nonlinearity. It is widely used to model nonlinear wave propagation in physical systems such as plasma, fluid mechanics, and optical fibers, where the effects of strong nonlinearity are significant.

The standard form of the mKdV equation is:

$$u_t - 6u^2u_x - u_{xxx} = 0 \tag{5.2.1}$$

- $u = u(x,t)$ is the wave amplitude,
- $u_t = \frac{\partial u}{\partial t}$ represents the temporal variation of the wave,
- $u_x = \frac{\partial u}{\partial x}$ is the spatial derivative,
- $u_{xxx} = \frac{\partial^3 u}{\partial x^3}$ represents the dispersion term.

- $-6u^2u_x$ is the cubic nonlinear term responsible for steepening the wave.

Characteristics of the mKdV Equation

1. **Cubic Nonlinearity:** Unlike the KdV equation (which has quadratic nonlinearity), the mKdV equation allows more complex wave interactions and supports both **soliton** and **kink-type solutions**.
2. **Dispersion:** The third-order derivative term balances the nonlinear effects, enabling stable wave propagation.
3. **Soliton Solutions:** The interplay between nonlinearity and dispersion allows the equation to admit **exact traveling wave solutions**, which maintain their shape and speed over time.

5.3 mKdV equation

We consider the $(1+1)$ dimensional mKdV Equation in the form [5.1

$$u_t - 6u^2u_x - u_{xxx} = 0 \tag{5.3.1}$$

Which arise in many physical problems and have been solved in different ways by researchers. We use the Sine Gordon expansion method and get some unique wave solution.

The moving coordinate is $u(x,t) = U(\xi)$ where $\xi = x - ct$ be the wave function. Using the transform we convert the equation [5.2] to ODE,

$$\therefore -cU'(\xi) - 6[U(\xi)]^2U'(\xi) + U'''(\xi) = 0 \tag{5.3.2}$$

Integrating it with respect to ξ and setting zero as the value of integrating constant, we acquire

$$-cU(\xi) - 2[U(\xi)]^3 + U" = 0 \tag{5.3.3}$$

According to the principal of balancing, we obtain $n + 2 = n + n + n$ implies that $n = 1$.

Now we can write Eq. (2.11) as

$$U(\omega) = B_1 \operatorname{Sin}(\omega)\, Cos(\omega) + A_1 Cos^2(\omega) + A_0 \tag{5.3.4}$$

We get a system of equation as bellow:

$\cos^3(\omega)$: $\quad -2{A_1}^3 + 6\,{B_1}^2 A_1 + 2A_1 = 0$

$\sin(\omega)\cos^2(\omega)$: $\quad -6B_1{A_1}^2 + 2{B_1}^3 + 2B_1 = 0$

$\cos^2(\omega)$: $\quad -6{A_1}^2 A_0 + 6A_0{B_1}^2 = 0$

$\sin(\omega)\cos(\omega)$: $\quad -12A_0A_1B_1 = 0$

$\cos(\omega)$: $\quad -6A_1{B_1}^2 - 6{A_0}^2 A_1 - cA_1 - 2A_1 = 0$

$\sin(\omega)$: $\quad -6{A_0}^2\, B_1 - 2{B_1}^3 - cB_1 - B_1 = 0$

$constant$: $\quad -2{A_0}^3 - 6A_0{B_1}^2 - cA_0 = 0$

After solving the above, we find the traveling wave equation $U(\xi)$ to Eq. [5.3.1] in the form of [2.10]

Case-I

$$c = -2, A_0 = 0, A_1 = 1, A_2 = 2, B_1 = 0, B_2 = B_2$$

which gives:

$$u(x,t) = U(\xi) = \tanh(\xi) \text{ with } \xi = x + 2t \qquad (5.3.5)$$

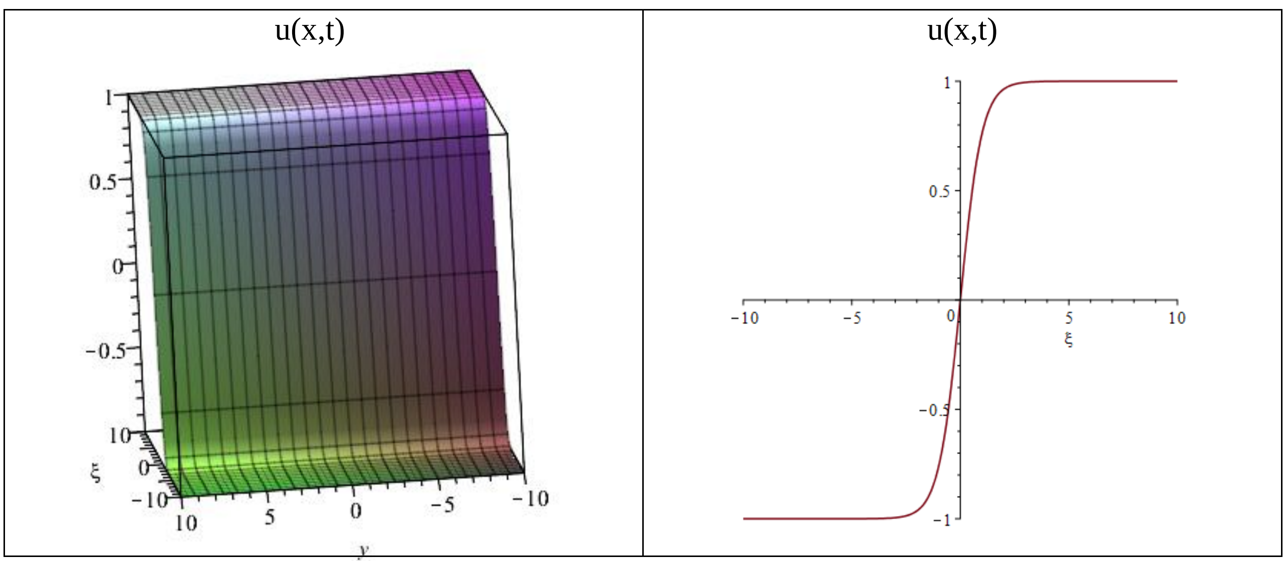

Figure 3: The graphical representation of Eq. [5.3.5] (3D on the left side and 2D on the right side)

Case-II

$$c = -2, A_0 = 0, A_1 = -1, A_2 = 2, B_1 = 0, B_2 = B_2$$

which gives:

$$u(x,t) = U(\xi) = -\tanh(\xi) \text{ with } \xi = x + 2t \tag{5.3.6}$$

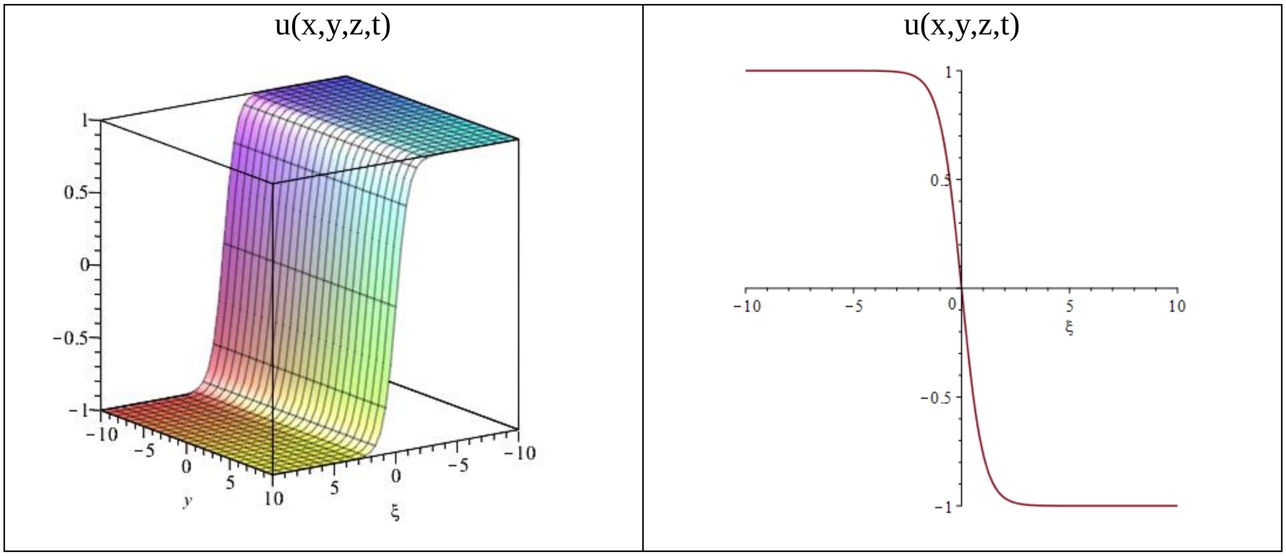

Figure 4: The graphical representation of Eq. [5.3.6] (3D on the left side and 2D on the right side)

5.4 Result and Discussion

For Case I, the modified Korteweg–de Vries (mKdV) equation provides a bright soliton solution obtained using the Sine-Gordon Expansion Method (SGEM). This solution corresponds to a localized solitary wave with positive amplitude that maintains its shape and velocity during propagation. The balance between cubic nonlinearity and dispersion enables the stability of this waveform, making it a fundamental example of nonlinear wave dynamics. The graphical representation (Fig. 3) in both 3D and 2D illustrates the robustness of the solution, showing a bell-shaped soliton that travels without distortion. The amplitude of the soliton is directly related to the system parameters, while its width decreases with increasing wave speed, producing narrower but more sharply peaked solitary structures. Physically, such bright soliton solutions have significant applications in plasma physics and optical fiber communication, where they describe localized energy packets traveling through nonlinear and dispersive media. These findings highlight the effectiveness of SGEM in systematically deriving exact solutions of the mKdV equation and confirm the physical relevance of bright solitons in modeling real-world nonlinear wave phenomena.

In Case II, the modified Korteweg–de Vries (mKdV) equation admits a dark soliton or kink-type solution, derived through the Sine-Gordon Expansion Method (SGEM). Unlike the bright soliton of Case I, this solution represents a step-like waveform that connects two distinct asymptotic states. The graphical representation (Fig. 4) in both 3D and 2D demonstrates the kink profile, where the wave undergoes a smooth but sharp transition between lower and higher states as it propagates along the spatial axis. The steepness of the transition depends on the wave speed and nonlinear parameters: increasing velocity produces sharper fronts, while dispersion tends to smooth the kink. Physically, such solutions are highly relevant in modeling domain walls in condensed matter physics, ion-acoustic waves in plasma, and nonlinear signal transmission in optical fibers. The results confirm that SGEM successfully captures both bright and dark soliton behaviors of the mKdV equation, emphasizing its versatility and effectiveness in deriving exact solutions of nonlinear dispersive systems.

Chapter VI

Conclusion and Future work

6.1 Conclusion

In this thesis, the **Sine-Gordon Expansion Method (SGEM)** has been successfully applied to the **Korteweg–de Vries (KdV) equation** and the **modified KdV (mKdV) equation** to obtain exact traveling wave solutions. The key outcomes of this study are:

1. **Exact Soliton Solutions:** Multiple types of soliton solutions, including solitary waves, periodic waves, and kink-type waves, were derived for both KdV and mKdV equations. These solutions demonstrate the balance between nonlinearity and dispersion inherent in these systems.
2. **Effectiveness of SGEM:** The Sine-Gordon Expansion Method proved to be a **systematic, reliable, and efficient analytical technique** for solving nonlinear evolution equations, capable of generating both known and previously undiscovered solutions.
3. **Physical Insight:** The obtained solutions provide a deeper understanding of nonlinear wave propagation, interactions, and stability in physical systems such as shallow water waves, plasma waves, and optical fibers.

Overall, this work highlights the utility of SGEM as a powerful tool for solving nonlinear PDEs, bridging the gap between theoretical modeling and practical applications in nonlinear dynamics.

6.2 Future Work

The findings of this thesis open several directions for future research:

1. **Extension to Other Nonlinear PDEs:** SGEM can be applied to more complex nonlinear equations, including higher-dimensional models, coupled systems, and equations with variable coefficients.
2. **Numerical Validation:** Combining analytical SGEM solutions with numerical simulations can validate solution behavior under varying initial and boundary conditions.

3. **Physical Applications:** The soliton and kink solutions can be used to model and analyze real-world phenomena in fluid mechanics, plasma physics, optical fibers, and other engineering systems.
4. **Generalization of Methods:** Development of hybrid analytical methods combining SGEM with other expansion techniques may allow the discovery of even more general and complex solutions.

This study lays a solid foundation for **further exploration of exact solutions in nonlinear dynamics** and demonstrates the versatility of the Sine-Gordon Expansion Method in modern applied mathematics and physics.

Reference

[1] ¨Ozlem Ersoy Hepson, Alper Korkmaz_, Kamyar Hosseini, Hadi Rezazadeh, Mostafa Eslami, “An Expansion Based on Sine-Gordon Equation to Solve KdV and modified KdV Equations in Confokrmable Fractional Forms”, Bol. Soc. Paran. Mat., 2022 (40), PP-1-10.

[2] M. Abul Kawser, M. Ali Akbar, Md. Ashrafuzzaman Khan, “k An investigation of the variable coefficients modified KdV equation arising in arterial mechanics by using two expansion techniques”, Result in physics, 2023, 106587

[3] Asim Zafar, “The expa function method and the conformable time-fractional KdV equations”, Nonlinear Engineering, 2019, 8; 728-732

[4] H. DEMIRAY, E. R. EL-ZAHAR, S. A. SHAN,” ON PROGRESSIVE WAVE SOLUTION FOR NON-PLANAR KDV EQUATION IN A PLASMA WITH q-NONEXTENSIVE ELECTRONS AND TWO OPPOSITELY CHARGED IONS”, TWMS J. App. and Eng. Math. 2020, V.10, N.2, pp. 532-546

[5] Qiuxia Xing · Zhiwei Wu · Dumitru Mihalache · Jingsong He, “Smooth positon solutions of the focusing modified Korteweg-de Vries equation”, arXiv:1705.06836v1 [nlin.SI] 19 May 2017

[6] Muath Awadalla1;*, Abdul Hamid Ganie2, Dowlath Fathima2, Adnan Khan3 and Jihan Alahmadi, “A mathematical fractional model of waves on Shallow water surfaces: The Korteweg-de Vries equation”, AIMS Mathematics, 2024, 9(5): 10561–10579.

[7] Lingfeng Xiao, Sirendaoerji, “The Modified Simple Equation Method and the Exact Solutions for the sine-Gordon Equation and the Generalized Variable-Coefficient KdV-mKdV Equation”, Advances in Applied Mathematics, 2016, 5(3), 443-449

[8] Yang Yang , Jian-ming Qi , Xue-hua Tang , and Yong-yi Gu,” Further Results about raveling Wave Exact Solutions of the (2+1)-Dimensional Modified KdV Equation”, Hindawi Advances in Mathematical Physics, 2019, 10

[9] A. S. Al-Fhaid, “New Exact Solutions for the Modified KdV-KP Equation Using the Extended F-Expansion Method”, Applied Mathematical Sciences, 2012, 6, 107, 5315 - 5332

[10] Ersoy Hepson O, Alper Korkmaz, K. Hosseini, Hadi Rezazadeh, “ An Expansion Based on Sine-Gordon Equation to Solve KdV and modified KdV Equations in Conformable Fractional Forms”, Scispace,

[11] Alharbi AR1, Almatrafi MB1 and Abdelrahman MAE1,2*, “An Extended Jacobian Elliptic Function Expansion Approach to the Generalized Fifth Order KdV Equation”, Journal of Physical Mathematics, 2019, 10:4

[12] Hossam A. Ghany and M. Zakarya, “Exp-function Method for Wick-type Stochastic Combined KdV-mKdV Equations”, Proceedings of Basic and Applied Sciences, 2013, Vol 1(1),

[13] Amna Iftikhar, Aysha Ghafoor, Tamour Zubair, Sahar Firdous and Syed Tauseef Mohyud-Din, “ (G´/G, 1/G)-expansion method for traveling wave solutions of (2+1) dimensional generalized KdV, Sin Gordon and Landau-Ginzburg-Higgs Equations”, 2013, Vol. 8(28), pp. 1349-1359,

[14] Julia Cen and Andreas Fring, “Complex solitons with real energies”, Mathematical and Theoretical, 2016, 49(36), 365202

[15] R. K. Alhefthi, “Solitary wave type solutions of nonlinear improved mKdV equation by modified techniques”, Revista Mexicana de F´ısica, 2024, 70 051301 1–13

[16] W. Djoudi & A. Zerarka, “Exact solutions for the KdV–mKdV equation with time-dependent coefficients using the modified functional variable method”, Cogent Mathematics, 2016, 3:1, 1218405

[17] Shuimeng Yu, “A Generalized (GG) -expansion Method and Its Application to the MKdV Equation”, International Journal of Nonlinear Science, 2009, Vol.8(2009) No.3,pp.374-378

[18] Dinh T Tran, Peter H van der Kamp and G RWQuispel, “Involutivity of integrals of sine-Gordon, modified KdV and potential KdV maps”, J. Phys. A: Math. Theor., 2010, 44 295206 (13pp)

[19] D. G. CRIGHTON, “Applications of KdV”, 1995, Acta Applicandae Mathematicae 39: 39-67,

[20] Yusuf Buba Chukkol and Mukhiddin Muminov “KINK WAVE SOLUTIONS TO KDV-BURGERS EQUATION WITH FORCING TERM”, Commun. Korean Math, 2020 35 No. 2, pp. 685{695

[21] Abdul-Majid Wazwaz, “Multiple Complex Soliton Solutions for the Integrable Sinh-Gordon and the Modified KdV-Sinh-Gordon Equation”, Applied Mathematics & Information Sciences, 2018, 12, No. 5, 899-905

[22] Yıldıray Keskin, İbrahim Çağlar and Ayşe Betül Koç, “Numerical Solution of Sine-Gordon Equation by Reduced Differential Transform Method”, World Congress on Engineering, 2011, 6 - 8, 2011

[23] Nizhum Rahman, “A note on the Sine-Gordon expansion method and its applications”, Da_odil International University Dhaka,1207,2022

[24] Jiajun Chen, Jianping Shi, Ao He & Hui Fang, “ Jiajun Chen, Jianping Shi, Ao He & Hui Fang”, scientific reports

[25] A. Sadighi, D.D. Ganji;_ and B. Ganjavi, “Traveling Wave Solutions of the Sine-Gordon and the Coupled Sine-Gordon Equations Using the Homotopy-Perturbation Method” Scientia Iranica, 2009, Vol. 16, No. 2, pp. 189{195

[26] Sidheswar Behera, “Optical soliton solutions to the Geophysical KdV equation using two robust analytical methods”, 2024

[27] Mukhtar Y. Y. ABDALLA a*, Bakery A. E. YOUNIES b, Haroon D. S. ADAM c, Ahmed M. I. ADAM d, Yagoub A. S. ARKO e, and Mohammad S. JAZMATI, “
MULTIPLE SOLITON SOLUTIONS OF THE MODEL mKdV WITH TIME-DEPENDENT VARIABLE COEFFICIENTS”, 2024, 28,6,5029-5036

[28] Zuntao Fu ∗, Shida Liu, Shikuo Liu, “New solutions to mKdV equation”, Elsevier, 2004, 326 , 364–374