Int. J. Appl. Comput. Math (2019) 5:57
https://doi.org/10.1007/s40819-019-0642-y

ORIGINAL PAPER

Extraction of Solitary Wave Features to the Heisenberg Ferromagnetic Spin Chain and the Complex Klein–Gordon Equations

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Published online: 19 April 2019

Abstract
In this paper, the modified Kudryashov method is applied for obtaining the exact travelling wave solutions for the (2 + 1)-dimensional Heisenberg ferromagnetic spin chain equation and the complex Klein–Gordon equation. The extracted solutions show distinct physical configurations. More precisely, the modified Kudryashov method is utilized for constructing the exact periodic and soliton solutions of these equations. As an outcome, in a broader sense, dark, bright, dark-bright, singular or combined singular and optical soliton solutions or wave features are derived from the equations. All solitons are verified through the corresponding equations with the help of Maple package program. The three-dimensional and two-dimensional waves are generated with the help of MATLAB for investigating the real significance of the proposed scheme to draw all solutions.

Keywords Heisenberg ferromagnetic spin chain equation · Complex Klein–Gordon equation · Modified Kudryashov method · New exact traveling wave solutions · Symbolic computation

Introduction

Over the few decades, the partial differential equations (PDEs) is playing a major role for solving the nonlinear evolution equations (NLEEs). It is widely accepted that NLEE plays an energetic role in finding exact solution for mathematical physics. Now-a-days, for the mathematicians and the physicists, finding the exact solution for NLEEs becomes an emerging research field. On the other side, nonlinear PDEs can be applied in explaining various important phenomena and many dynamic processes in chemistry, mechanics, biology, and

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physics. Many possible solutions to NLEEs, some special form of solutions, for example solitons, is dependent on a single combination of variables. Soliton refers to self-reinforcing solitary wave, a wave packet or pulse which maintains its shape at the time of its travel at constant speed. *John Scott Russel* (1808–1882) introduced the term soliton fist when he observed the solitary waves at the Union Canal, Scotland [1]. Many methods are proposed in the literature for acquiring the traveling wave solutions of NLEEs, such as the Backlund transformation method [2], the inverse scattering method [3], the Jacobi elliptic function method [4], the Wroanskian technique [5], tanh/coth method [6, 7], sine/cosine method [8], the exponential function method [9], the first integral method [10], the homogeneous balance method [11, 12], the auxiliary equation method [13–15], the Darboux transformation method [16, 17], the modified simple equation method [18, 19], the modified extended tanh-function method [20], the $(\acute{G}/G)$-expansion method [21–29], the $\exp(-\phi(\xi))$-expansion method [30, 31], the homotopy perturbation method [32–36], the F-expansion method [37–39], the variational iteration method [40], the conformable time-fractional method [41, 42], the improved tanϕη/2-expansion method [43], the direct algebraic method [44], the extended direct algebraic method [45–47], the multiple scales method [48], the amplitude Ansatz method [49], and the auxiliary equation method [50].

The (2 + 1)-dimensional Heisenberg ferromagnetic spin chain (HFSC) equation is described as

$$iq_t + \alpha q_{xx} + \mu q_{yy} + \delta q_{xy} - \gamma q|q|^2 = 0, \quad i = \sqrt{-1} \tag{1}$$

where $\alpha = \sigma^4(J + J_2),\ \mu = \sigma^4(J_1 + J_2),\ \delta = 2\sigma^4 J_2, \gamma = 2\sigma^4 A$.

Here, x, y and t are independent variables and $q(x, t, y)$ is the dependent variable in Eq. (1). The parameter σ describes the lattice parameter, the bilinear exchange interactions' coefficients along X and Y axes are represented by J and J_1 respectively. The neighboring interaction on the diagonal is denoted by J_2 while the uniaxial crystal field anisotropy parameter is denoted by A [51]. The HFSC equation drawed attention of a large number of researchers [52, 53]. Many nonlinear models play significant roles in the field of fiber optics, other fields in mathematical physics and Soliton theory.

On the other side, the complex nonlinear Klein–Gordon equation:

$$u_{tt} - p^2 u_{xx} + qu + r|u|^2 u = 0 \tag{2}$$

where $pqr \neq 0$. The Klein-Gordon equation in (2) is utilized in explaining many mathematical physics and physical problems, such as problems in optical fibers, phase transition, crystal growths, dislocations etc. Many mathematical physics and physical problems can also be described using the Sine–Gordon and the Double-Sine–Gordon equations [54–56] which play vital roles for exploring the solitons and condensed matter physics [54], in finding the solitons' interactions in a collisionless plasma, the recurrence of initial stage, and in investigating the nonlinear wave equations [55]. The exact solutions of nonlinear PDEs are obtained by applying a new solution method, namely the modified Kudryashov method [57] to solve problems in mathematical physics. As a result, the technique received significant attention to the researchers for its capabilities in obtaining exact solutions of PDEs, for example, wave solutions for the Tzitzéica-type non-linear evolution equations [58], solutions for the time-fractional KdV–KZD equation [59], solutions for the time-fractional modified Sawada–Kotera equation [60], analytical exact solutions of the time-fractional generalized Fisher equation [61]. In this paper, by applying the modified Kudryashov method, we find out some new solutions of the (2 + 1)-D Heisenberg ferromagnetic Spin chain equation and the complex klein–Gordon equation. Wave features, such as the Dark, bright, dark-bright, singular or combined singular and optical soliton solutions are derived.

Outline of the Modified Kudryashov Method

To extract the new exact solution of nonlinear differential equations, the modified Kudryashov method is considered as a robust problem-solving technique. A brief introduction of the modified Kudryashov method [62] is presented here for obtaining thenew exact solutionsfor a given nonlinear PDEs. In order to achieve the solution, let us consider a nonlinear PDE

$$F(v, v_x, v_t, v_{xx}, v_{xt}, v_{tt}, \ldots) = 0, \tag{3}$$

where the function $v = v(x, t)$ is unknown and F is a polynomial. Major steps are summerized as:

Step 1 At first, a transformation $v(x, t) = V(\xi)$ is introduced, where $\xi = kx + \omega t$, varies according to Eq. (3). Then the equation is reduced to the following nonlinear ODE

$$P\big(V, V', V'', \ldots\big) = 0, \tag{4}$$

Here, P is a polynomial in V and its derivatives. Moreover, the superscripts represent the ordinary derivatives with respect to ξ.

Step 2 The nonlinear Eq. (4) can be solved as

$$V(\xi) = A_0 + \sum_{i=1}^{N} A_i Q^i(\xi), \quad A_N \neq 0, \tag{5}$$

in which the constants $A_i (i = 0, 1 \ldots N)$ are calculated later anda positive integer N is determined from the balancing principle on Eq. (5). $Q(\xi)$ satisfies the following equation

$$Q'(\xi) = (Q^2(\xi) - Q(\xi)) \ln(A), \tag{6}$$

with the exact solution $Q(\xi) = \frac{1}{1+dA^{\xi}}$, where $A \neq 0, 1$.

Step 3 Substituting Eqs. (5) and (6) in Eq. (4) and after few mathematical operations, a system of algebraic equations is obtained in parameters $A_i (i = 0, 1 \ldots N)$, k, and ω. We can finally generate the new exact solutions for the Eq. (3) by setting the extracted values in Eq. (5).

Mathematical Analysis

Solution of the Nonlinear (2 + 1)-Dimensional Heisenberg Ferromagnetic Spin Chain Equation

The following transformation is employed:

$$\psi(x, t, y) = U(\xi) e^{i\eta}, \quad \xi = (x + y - vt), \quad \eta = -k_1 x - k_2 y + \omega t \tag{7}$$

By employing the transformation Eq. (7) into the nonlinear (2 + 1)-dimensional Heisenberg ferromagnetic spin chain equation Eq. (1) is transformed into following nonlinear ordinary differential equation

$$\big(-\omega - \alpha k_1^2 - \delta k_1 k_2 - \mu k_2^2\big) U - \gamma U^3 + (\alpha + \delta + \mu) U'' = 0 \tag{8}$$

$N = 1$ is obtained using the homogeneous balance principle. Then, the solution of Eq. (8) becomes

$$U(\xi) = A_0 + A_1 Q(\xi) \tag{9}$$

By substituting Eq. (9) along with its second derivative into Eq. (8) and equating the like powers of $Q(\xi)$, the following system of equations is obtained:

$$-\gamma A_1^3 + 2(\alpha + \beta + \mu)A_1 \ln(A)^2 = 0$$

$$-3\gamma A_0 A_1^2 - 3(\alpha + \beta + \mu)A_1 \ln(A)^2 = 0$$

$$-\left(\alpha k_1^2 + \delta k_1 k_2 + \mu k_2^2 + \omega\right)A_1 - 3\gamma A_0^2 A_1 + (\alpha + \beta + \mu)A_1 \ln(A)^2 = 0$$

$$-\left(\alpha k_1^2 + \delta k_1 k_2 + \mu k_2^2 + \omega\right)A_0 - \gamma A_0^3 = 0$$

By solving the above system the following different solution sets are extracted:

Set-I $\omega = -\frac{1}{2}\ln(A)^2\alpha - \frac{1}{2}\ln(A)^2\delta - \frac{1}{2}\ln(A)^2\mu - \alpha k_1^2 - \delta k_1 k_2 - \mu k_2^2$, $A_0 = \frac{1}{2}\frac{\sqrt{2}\sqrt{\gamma(\alpha+\delta+\mu)}\ln(A)}{\gamma}$ and $A_1 = -\frac{\ln(A)(\alpha+\delta+\mu)\sqrt{2}}{\sqrt{\gamma(\alpha+\delta+\mu)}}$

Set-I resembles the following exact solution of the nonlinear (2 + 1)-D Heisenberg ferromagnetic spin chain equations derived:

$$\psi_1(x,t,y) = \left(\frac{1}{2}\frac{\sqrt{2}\sqrt{\gamma(\alpha+\delta+\mu)}\ln(A)}{\gamma} - \frac{\ln(A)(\alpha+\delta+\mu)\sqrt{2}}{\sqrt{\gamma(\alpha+\delta+\mu)}\left(1 + dA^{-t(-2\alpha k_1 - \delta k_1 - \delta k_2 - 2\mu k_2) + x + y}\right)}\right) \times e^{i\left(\left(-\frac{1}{2}\ln(A)^2\alpha - \frac{1}{2}\ln(A)^2\delta - \frac{1}{2}\ln(A)^2\mu - \alpha k_1^2 - \delta k_1 k_2 - \mu k_2^2\right)t - k_1 x - k_2 y\right)} \tag{10}$$

Set-II $w = -\frac{1}{2}\ln(A)^2 a\alpha^2 - \frac{1}{2}\ln(A)^2 a\beta^2 - ap^2 - aq^2$, $A_0 = -\frac{\sqrt{-(pb_1+qb_2)a(\alpha^2+\beta^2)}\ln(A)}{2pb_1+2qb_2}$ and $A_1 = -\frac{a\ln(A)(\alpha^2+\beta^2)}{\sqrt{-(pb_1+qb_2)a(\alpha^2+\beta^2)}}$.

Set-II resembles the following exact solution of the nonlinear (2 + 1)-D Heisenberg ferromagnetic spin chain equations explored:

$$\psi_1(x,t,y) = \left(-\frac{1}{2}\frac{\sqrt{2}\sqrt{\gamma(\alpha+\delta+\mu)}\ln(A)}{\gamma} + \frac{\ln(A)(\alpha+\delta+\mu)\sqrt{2}}{\sqrt{\gamma(\alpha+\delta+\mu)}\left(1 + dA^{-t(-2\alpha k_1 - \delta k_1 - \delta k_2 - 2\mu k_2) + x + y}\right)}\right) \times e^{i\left(\left(-\frac{1}{2}\ln(A)^2\alpha - \frac{1}{2}\ln(A)^2\delta - \frac{1}{2}\ln(A)^2\mu - \alpha k_1^2 - \delta k_1 k_2 - \mu k_2^2\right)t - k_1 x - k_2 y\right)} \tag{11}$$

The three dimensional (3D) and two dimensional (2D) plots of the Eqs. (10) and (11), have been demonstrated in Figs. 1 and 2, respectively.

Solution of the Nonlinear Complex Klein–Gordon Equation

By considering the following transformation:

$$\psi(x,t) = U(\xi)e^{i\eta}, \quad \xi = k(x - at), \quad \eta = ax + \omega t \tag{12}$$

where k, a, ω are constants. By utilizing the transformation of Eq. (12) into the complex Klein–Gordon Eq. (2) is reduced to the following nonlinear ODE:

$$k^2(a^2 - p^2)U'' - 2iak(\omega + p^2)U' + (a^2p^2 + q - \omega^2)U + rU^3 = 0 \tag{13}$$

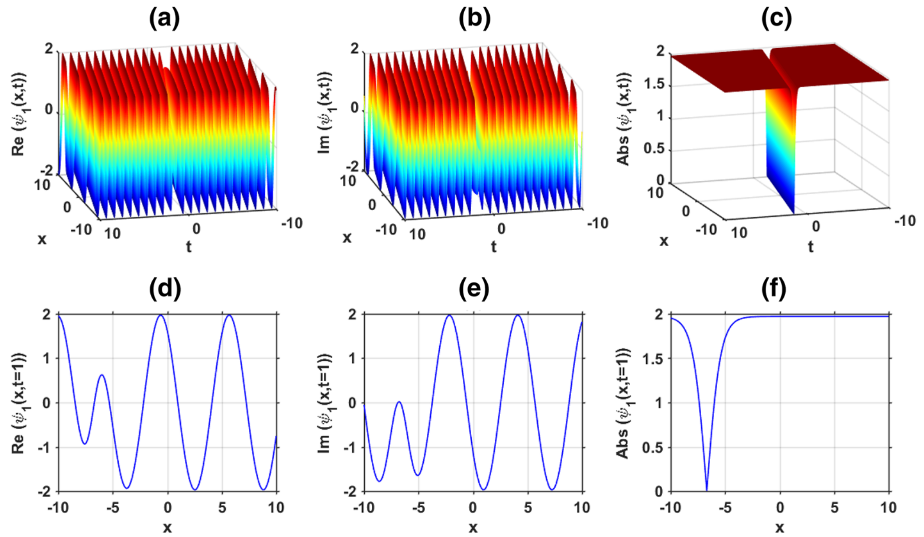

Fig. 1 **a**–**c** 3D illustrations of the Eq. (10) for the choosing arbitrary parameters $\alpha = 1, \mu = 1, \delta = 1, \gamma = 1, A = 1, k_1 = 1, k_2 = 1, d = 3, y = 0$ within $-10 \leq x \leq 10, -10 \leq t \leq 10$, and **d**–**f** 2D line illustrations of **a**–**c** at $y = 0, t = 0$, respectively

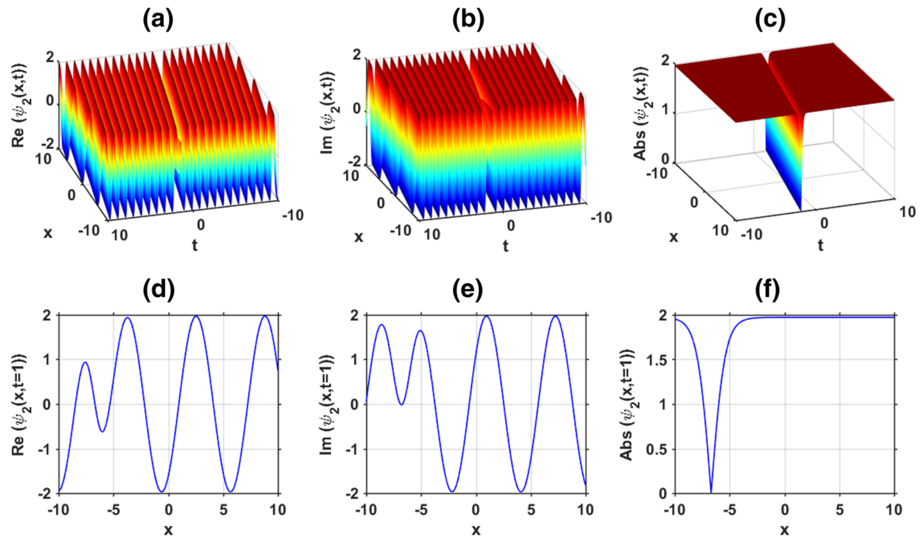

Fig. 2 **a**–**c** 3D illustrations of the Eq. (11) for the choosing arbitrary parameters $\alpha = 1, \mu = 1, \delta = 1, \gamma = 1, A = 1, k_1 = 1, k_2 = 1, d = 3, y = 0$ within $-10 \leq x \leq 10, -10 \leq t \leq 10$, and **d**–**f** 2D line illustrations of **a**–**c** at $y = 0, t = 0$, respectively

By utilizing the homogeneous balance principle, we get $m + 2 = 3m$, hence $m = 1$.Then, the solution of the Eq. (8) becomes

$$U(\xi) = A_0 + A_1 Q(\xi) \tag{14}$$

By substituting Eq. (14) along with its second derivative into Eq. (13) and equating the like powers of $Q(\xi)$, then we get in the following system of equations:

$$2\ln(A)^2a^2k^2A_1 - 2\ln(A)^2k^2p^2A_1 + 7A_1^3 = 0,$$

$$-3\ln(A)^2a^2k^2A_1 + 3\ln(A)^2k^2p^2A_1 + 21A_0A_1^2 = 0,$$

$$\ln(A)^2a^2k^2A_1 - \ln(A)^2k^2p^2A_1 + a^2p^2A_1 - p^4A_1 + 21A_0^2A_1 + qA_1 = 0,$$

$$a^2p^2A_0 - p^4A_0 + 7A_0^3 + qA_0 = 0.$$

By solving the above system, we receive the following different solution sets:

Set-I $a = \frac{\sqrt{(k^2\ln(A)^2-2p^2)(\ln(A)^2k^2p^2-2p^4+2q)}}{k^2\ln(A)^2-2p^2}$, $A_0 = \frac{1}{7}\frac{\sqrt{-(7k^2\ln(A)^2-14p^2)q}\,\ln(A)k}{k^2\ln(A)^2-2p^2}$ and $A_1 = \frac{2kq\ln(A)}{\sqrt{-(7k^2\ln(A)^2-14p^2)q}}$.

Set-I possesses the following exact solution of the complex Klein–Gordon equations extracted:

$$\psi_1(x,t) = \left(\frac{1}{7}\frac{\sqrt{-\left(7k^2ln(A)^2-14p^2\right)q}ln(A)k}{k^2ln(A)^2-2p^2} + \frac{2kqln(A)}{\sqrt{-\left(7k^2ln(A)^2-14p^2\right)q}\left(1+dA^{k\left(-\frac{\sqrt{\left(k^2ln(A)^2-2p^2\right)\left(k^2ln(A)^2p^2-2p^4+2q\right)}t}{k^2ln(A)^2-2p^2}+x\right)}\right)}\right) \times e^{i\left(-p^2t+\frac{\sqrt{\left(k^2ln(A)^2-2p^2\right)\left(k^2ln(A)^2p^2-2p^4+2q\right)}x}{k^2ln(A)^2-2p^2}\right)} \tag{15}$$

Set-II $a = -\frac{\sqrt{(k^2\ln(A)^2-2p^2)(\ln(A)^2k^2p^2-2p^4+2q)}}{k^2\ln(A)^2-2p^2}$, $A_0 = \frac{1}{7}\frac{\sqrt{-(7k^2\ln(A)^2-14p^2)q}\,\ln(A)k}{k^2\ln(A)^2-2p^2}$ and $A_1 = \frac{2kq\ln(A)}{\sqrt{-(7k^2\ln(A)^2-14p^2)q}}$

Set-II possesses the following exact solution of the complex Klein–Gordon equations extracted:

$$\psi_2(x,t) = \left(\frac{1}{7}\frac{\sqrt{-\left(7k^2ln(A)^2-14p^2\right)q}ln(A)k}{k^2ln(A)^2-2p^2} + \frac{2kqln(A)}{\sqrt{-\left(7k^2ln(A)^2-14p^2\right)q}\left(1+dA^{k\left(\frac{\sqrt{\left(k^2ln(A)^2-2p^2\right)\left(k^2ln(A)^2p^2-2p^4+2q\right)}t}{k^2ln(A)^2-2p^2}+x\right)}\right)}\right) \times e^{i\left(-p^2t-\frac{\sqrt{\left(k^2ln(A)^2-2p^2\right)\left(k^2ln(A)^2p^2-2p^4+2q\right)}x}{k^2ln(A)^2-2p^2}\right)} \tag{16}$$

Set-III $a = \frac{\sqrt{(k^2\ln(A)^2-2p^2)(\ln(A)^2k^2p^2-2p^4+2q)}}{k^2\ln(A)^2-2p^2}$, $A_0 = -\frac{1}{7}\frac{\sqrt{-(7k^2\ln(A)^2-14p^2)q}\,\ln(A)k}{k^2\ln(A)^2-2p^2}$ and $A_1 = -\frac{2kq\ln(A)}{\sqrt{-(7k^2\ln(A)^2-14p^2)q}}$.

Set-III possesses the following exact solution of the complex Klein–Gordon equations extracted:

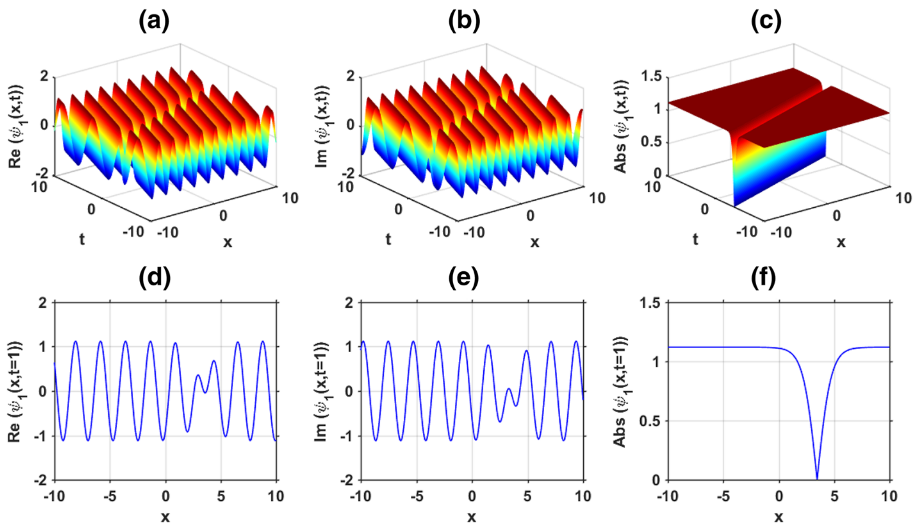

Fig. 3 **a**–**c** 3D illustrations of the Eq. (15) for the choosing arbitrary parameters $p = 1, A = 5, k = -1, d = 3, q = 2$ within $-10 \leq x \leq 10, -10 \leq t \leq 10$, and **d**–**f** 2D line illustrations of **a**–**c** at $t = 0$, respectively

$$\psi_3(x,t) = \left(-\frac{1}{7}\frac{\sqrt{-(7k^2 ln(A)^2 - 14p^2)q}\, ln(A)k}{k^2 ln(A)^2 - 2p^2} - \frac{2kq ln(A)}{\sqrt{-(7k^2 ln(A)^2 - 14p^2)q}\left(1 + dA^{k\left(-\frac{\sqrt{(k^2 ln(A)^2 - 2p^2)(k^2 ln(A)^2 p^2 - 2p^4 + 2q)}\,t}{k^2 ln(A)^2 - 2p^2} + x\right)}\right)} \right) \times e^{i\left(-p^2 t - \frac{\sqrt{(k^2 ln(A)^2 - 2p^2)(k^2 ln(A)^2 p^2 - 2p^4 + 2q)}\,x}{k^2 ln(A)^2 - 2p^2}\right)} \tag{17}$$

Set-VI $a = -\frac{\sqrt{(k^2 \ln(A)^2 - 2p^2)(\ln(A)^2 k^2 p^2 - 2p^4 + 2q)}}{k^2 \ln(A)^2 - 2p^2}$, $A_0 = -\frac{1}{7}\frac{\sqrt{-(7k^2 \ln(A)^2 - 14p^2)q}\,\ln(A)k}{k^2 \ln(A)^2 - 2p^2}$ and $A_1 = -\frac{2kq\ln(A)}{\sqrt{-(7k^2\ln(A)^2 - 14p^2)q}}$

Set-IV possesses the following exact solution of the complex Klein–Gordon equations extracted:

$$\psi_4(x,t) = \left(-\frac{1}{7}\frac{\sqrt{-(7k^2 ln(A)^2 - 14p^2)q}\, ln(A)k}{k^2 ln(A)^2 - 2p^2} - \frac{2kq ln(A)}{\sqrt{-(7k^2 ln(A)^2 - 14p^2)q}\left(1 + dA^{k\left(\frac{\sqrt{(k^2 ln(A)^2 - 2p^2)(k^2 ln(A)^2 p^2 - 2p^4 + 2q)}\,t}{k^2 ln(A)^2 - 2p^2} + x\right)}\right)} \right) \times e^{i\left(-p^2 t - \frac{\sqrt{(k^2 ln(A)^2 - 2p^2)(k^2 ln(A)^2 p^2 - 2p^4 + 2q)}\,x}{k^2 ln(A)^2 - 2p^2}\right)} \tag{18}$$

The three dimensional (3D) and two dimensional (2D) plots of the Eqs. (15), (16), (17) and (18) have been demonstrated in Figs. 3, 4, 5 and 6 respectively.

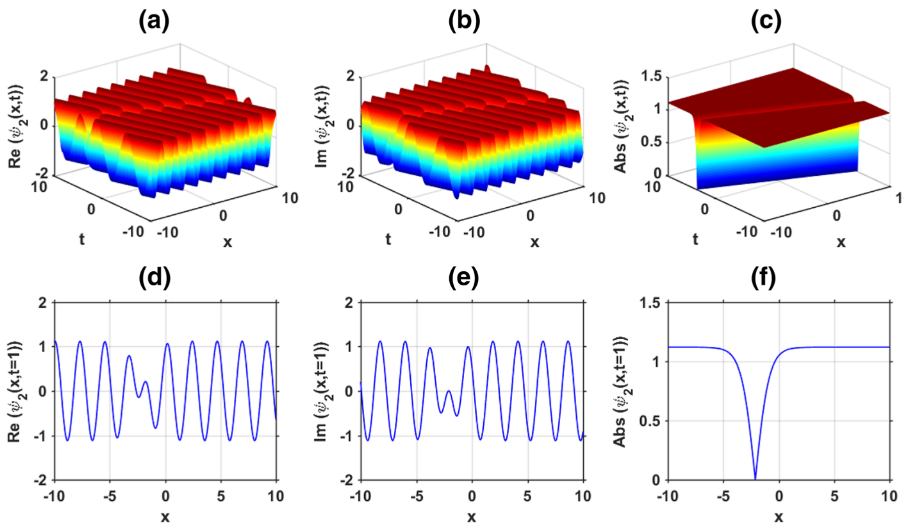

Fig. 4 **a**–**c** 3D illustrations of the Eq. (16) for the choosing arbitrary parameters $p = 1, A = 5, k = -1, d = 3, q = 2$ within $-10 \leq x \leq 10, -10 \leq t \leq 10$, and **d**–**f** 2D line illustrations of **a**–**c** at $t = 0$, respectively

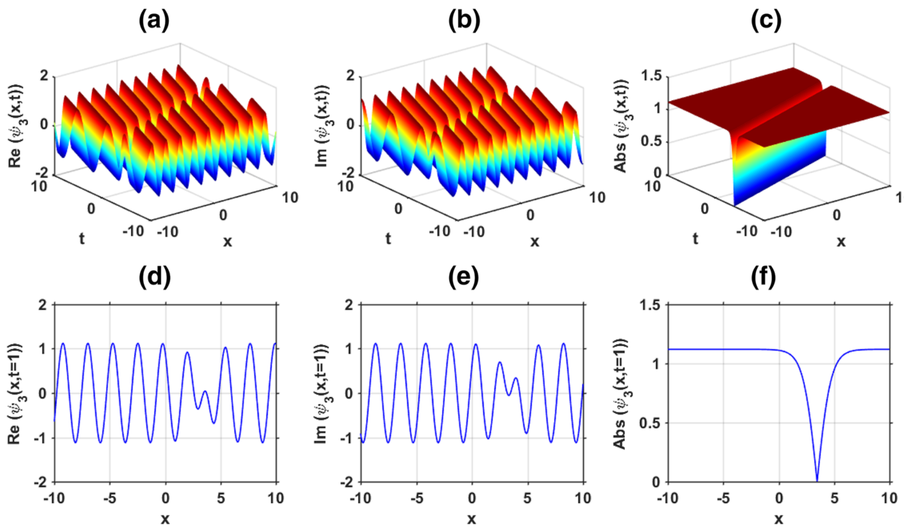

Fig. 5 **a**–**c** 3D illustrations of the Eq. (17) for the choosing arbitrary parameters $p = 1, A = 5, k = -1, d = 3, q = 2$ within $-10 \leq x \leq 10, -10 \leq t \leq 10$, and **d**–**f** 2D line illustrations of **a**–**c** at $t = 0$, respectively

Conclusions

The modified Kudryashov method was applied efficiently to establish the exact traveling wave solutions to the (2 + 1)-dimensional Heisenberg ferromagnetic spin chain equation and the complex Klein–Gordon equation. The objective of this paper was to extract the new exact solutions specifically dark, bright, dark-bright, singular or combined singular solution other soliton solutions. It was also demonstrated in the paper that the proposed method was

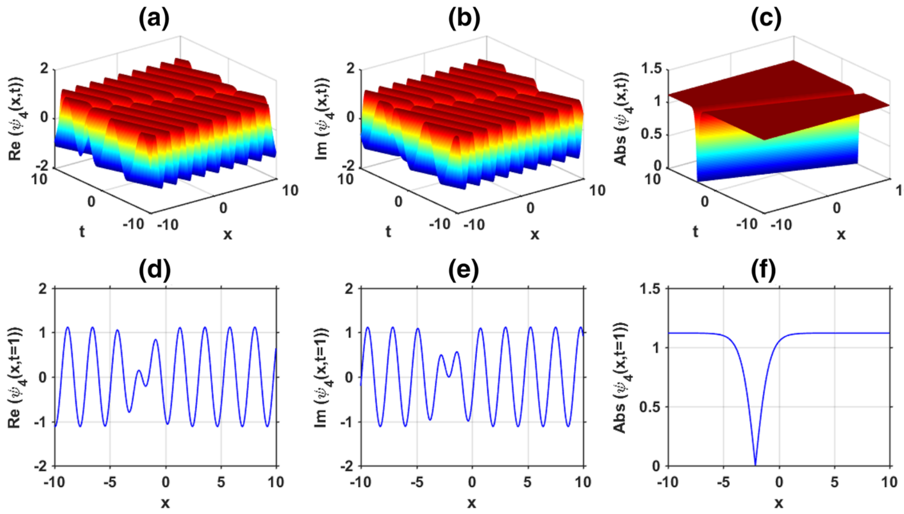

Fig. 6 **a**–**c** 3D illustrations of the Eq. (18) for the choosing arbitrary parameters $p = 1, A = 5, k = -1, d = 3, q = 2$ within $-10 \leq x \leq 10, -10 \leq t \leq 10$, and **d**–**f** 2D line illustrations of **a**–**c** at $t = 0$, respectively

quite effective in finding exact solutions of the nonlinear evolution equations. Moreover, the proposed method is straightforward and could be applied to many other the nonlinear evolution equations. It is worthy of further study and might have substantial physics and engineering.

Acknowledgements The authors would like to thank Mohammed Khurshed Alam of department of Electrical and Electronic Engineering, Uttara University for his valuable suggestions in preparing the manuscript.

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