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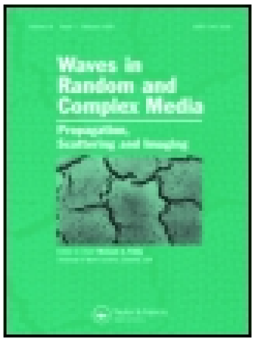
3 authors, including:

S. M. Rayhanul Islam
Pabna University of Science and Technology

K.M. Abdul Al Woadud
Uttara University



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Analytical studies on the Benney–Luke equation in mathematical physics

S. M. Rayhanul Islam[a], Kamruzzaman Khan[a] and K. M. Abdul Al Woadud[b]

[a]Department of Mathematics, Pabna University of Science and Technology, Pabna, Bangladesh; [b]Department of Electrical & Electronic Engineering, Uttara University, Dhaka, Bangladesh

ABSTRACT
The enhanced (G'/G)-expansion method presents wide applicability to handling nonlinear wave equations. In this article, we find the new exact traveling wave solutions of the Benney–Luke equation by using the enhanced (G'/G)-expansion method. This method is a useful, reliable, and concise method to easily solve the nonlinear evaluation equations (NLEEs). The traveling wave solutions have expressed in term of the hyperbolic and trigonometric functions. We also have plotted the 2D and 3D graphics of some analytical solutions obtained in this paper.

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1. Introduction

The exploration of exact traveling wave solutions to nonlinear evolution equation plays an important role in the study of nonlinear physical phenomena for various fields of science and engineering, especially in mathematical physics, plasma physics, fluid dynamics, propagation of shallow water waves, and so on. The Benney–Luke equation deals with the derivative of the basic shallow water equations from the three-dimensional Euler equation for the irrotational and incompressible fluid. Those equations include the Benney–Luke equation which is obtained under the assumptions of small amplitude and long wave. The Benney–Luke equation is as follows:

$$u_{tt} - u_{xx} + \alpha\, u_{xxxx} - \beta\, u_{xxtt} + u_t u_{xx} + 2u_x u_{xt} = 0, \tag{1.1}$$

where α and β are positive numbers such that $\alpha - \beta = \sigma - 1/3$, σ is the Bond number, which measure the effects of surface tension and is a formally valid approximation for describing two ways water wave propagation in the presence of surface tension [1]. Pego and Quintero [2] considered the proliferation of long water waves with small amplitude. They presented that in the manifestation of an exterior stiffness, the circulation of such waves is manages by Benney–Luke equation, originally derived by Benney and Luke [3].

In recent year, many explicit solutions of nonlinear evolution equations by using various different methods is the main goal for many researches and many powerful methods have been established and developed such as the Hirota's bilinear [4,5], the trigonometric function

CONTACT S. M. Rayhanul Islam rayhanul_math@pust.ac.bd, rayhanulmath@yahoo.com

series method [6], the modified mapping method [7], the modified trigonometric function series method [8,9], the dynamical systems approach and the bifurcation method [10,12,13], the modified (G'/G)-expansion method [11], the extended (G'/G)-expansion method [14], the new method [15], the infinite series method and Jacobi elliptic function expansion method [16], the tanh-coth expansion method and the Jacobi elliptic function expansion method [17], the First integral method [18], the auxiliary ordinary differential equation method [19], the Jacobi elliptic function expansion method [20], the generalized (G'/G)-expansion method [21], the extended tanh method [22,23], the Exp-function method [24], the extended F- expansion method [25], the F- expansion method [26], the (G'/G)-expansion method [27–29], the modified simple equation method [30], the enhanced (G'/G)-expansion method [31,32], the exp(-Φ(ξ))-expansion method [33–35], and so on.

The objective of this article is to implement the infliction of the enhanced (G'/G)-expansion method to construct the exact traveling wave solutions for NLEEs in mathematical physics via the Benney-Luke equation.

The article is prepared as follows: In Section 2, the enhanced (G'/G)-expansion method has been discussed. In Section 3, we apply this method to the nonlinear evolution equations pointed out above. In Section 4, comparison, in Section 5, Discussion and in Section 6 conclusion are given.

2. Enhanced (G'/G)-expansion method

In this section, we describe in detail the enhanced (G'/G)-expansion method for finding traveling wave equations of nonlinear equations. Any nonlinear equation in two independent variables x and t can be expressed in following form:

$$\Psi(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt} \ldots) = 0, \tag{2.1}$$

where $u(\xi) = u(x, t)$ is an unknown function, Ψ is a polynomial of $u(x,t)$ and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. The following steps are involved in finding the solution of nonlinear Equation (2.1) using this method.

Step 1: The given PDE (2.1) can be transformed into ODE using the transformation $\xi = x \pm \omega t$, where ω is the speed of traveling wave such that $\omega \in R - \{0\}$.

The traveling wave transformation permits us to reduce Equation (2.1) to the following ODE:

$$\Psi(u, u', u'', \ldots) = 0, \tag{2.2}$$

where Ψ is a polynomial in $u(\xi)$ and its derivatives, where $u'(\xi) = \frac{du}{d\xi}$, $u''(\xi) = \frac{d^2u}{d\xi^2}$, and so on.

Step 2: Now we suppose that the Equation (2.2) has a general solution of the form

$$u(\xi) = \sum_{i=-n}^{n} \left(\frac{a_i (G'/G)^i}{\left(1 + \lambda (G'/G)\right)^i} + b_i (G'/G)^{i-1} \sqrt{\sigma \left(1 + \frac{(G'/G)^2}{\mu}\right)} \right), \tag{2.3}$$

Subject to the condition that $G = G(\xi)$, satisfy the equation

$$G'' + \lambda G = 0, \tag{2.4}$$

where a_i, $b_i (-n \le i \le n; n \in N)$, and λ are constant to be determined, provided that $\sigma = \pm 1$ and $\mu \ne 0$.

Step 3: The positive integer n can be determined by balancing the highest order derivatives to the highest order nonlinear terms that appear in Equation (2.1) or in Equation (2.2). More precisely, we define the degree of $u(\xi)$ as $D(u(\xi)) = n$ which gives rise to the degree of other expression as follows:

$$D\left(\frac{d^q u}{d\xi^q}\right) = n + q,\ D\left(u^p\left(\frac{d^q u}{d\xi^q}\right)^s\right) = np + s\,(n+q), \tag{2.5}$$

Step 4: We substitute Equation (2.3) into Equation (2.2) and use Equation (2.4). We then collect all the coefficient of $(G'/G)^j$ and $(G'/G)^j \sqrt{\sigma\left(1 + \frac{(G'/G)^2}{\mu}\right)}$ together. Since Equation (2.3) is a solution of Equation (2.2). we can set each of the coefficient equal to zero which leads to a system of algebraic equations in terms of $a_i, b_i(-n \leq i \leq n; n \in N)$, λ and ω. One can solves easily these system equations using Maple.

Step 5: For $\mu < 0$ general solution of Equation (2.4) gives

$$\frac{G'}{G} = \sqrt{-\mu}\ \tanh\left(A + \sqrt{-\mu}\ \xi\right), \tag{2.6}$$

and

$$\frac{G'}{G} = \sqrt{-\mu}\ \coth\left(A + \sqrt{-\mu}\ \xi\right), \tag{2.7}$$

And for $\mu > 0$, we get

$$\frac{G'}{G} = \sqrt{\mu}\ \tan\left(A - \sqrt{\mu}\ \xi\right), \tag{2.8}$$

and

$$\frac{G'}{G} = \sqrt{\mu}\ \cot\left(A + \sqrt{\mu}\ \xi\right), \tag{2.9}$$

where A is an arbitrary constant. Finally we can construct a number of families of traveling wave solutions of Equation (2.1) by substituting the values of $a_i, b_i(-n \leq i \leq n; n \in N)$, $\lambda$and ω(obtained in Step 3) and using Equations (2.6)–(2.9) into Equation (2.3).

3. Application

In order to solve the Equation (1.1) by the enhanced (G'/G)-expansion method, now, we utilize the wave variable $u\,(x,t) = u\,(\xi)$, $\xi = x - V\,t$, where V is the speed of traveling, then the Equation (1.1) reduces to a ordinary differential equation (ODE) as

$$(V^2 - 1)\,u'' + (\alpha - \beta\,V^2)\,u^{(4)} - 3\,Vu'u'' = 0, \tag{3.1}$$

Equation (3.1) is integrable, therefore integrating with respect to ξ once and choosing the integration constant to zero, we obtain

$$(V^2 - 1)\,u' + (\alpha - \beta\,V^2)\,u^3 - \frac{3}{2}\,Vu'^2 = 0, \tag{3.2}$$

Taking the homogeneous balance between the highest-order derivative u'^2 and the nonlinear term of the highest order u^3, we find $N = 1$. Using the assumptions of the enhanced (G'/G)-expansion method (2.3)–(2.5) gives the solution in the form

$$\begin{aligned} u(\xi) &= a_0 + \frac{a_1(G'/G)}{1+\lambda(G'/G)} + \frac{a_{-1}(1+\lambda(G'/G))}{(G'/G)} + b_0(G'/G)^{-1}\sqrt{\sigma\left(1+\frac{(G'/G)^2}{\mu}\right)} \\ &+ b_1\sqrt{\sigma\left(1+\frac{(G'/G)^2}{\mu}\right)} + b_{-1}(G'/G)^{-2}\sqrt{\sigma\left(1+\frac{(G'/G)^2}{\mu}\right)}, \end{aligned} \tag{3.3}$$

where $G = G(\xi)$ satisfies Equation (2.4). Substituting Equation (3.3) into Equation (3.2) and using Equation (2.4), we get a polynomial in $(G'/G)^j$ and $(G'/G)^j\sqrt{\sigma\left(1+\frac{(G'/G)^2}{\mu}\right)}$. Setting the coefficient of $(G'/G)^j$ and $(G'/G)^j\sqrt{\sigma\left(1+\frac{(G'/G)^2}{\mu}\right)}$ equal to zero, we obtain a system containing a large number of algebraic equations in terms of unknown coefficients. We have solved this system of equations by using mathematical software Maple 13 and obtained the following set of solutions:

Set 1: $V = \pm\sqrt{\frac{1+4\alpha\mu}{1+4\beta\mu}}, a_{-1} = 0, a_0 = a_0,\ a_1 = \pm\frac{4(\alpha\mu\lambda^2-\beta\mu\lambda^2-\beta+\alpha)}{\sqrt{1+4\beta\mu}\sqrt{1+4\alpha\mu}}, b_0 = 0, b_1 = 0, b_{-1} = 0,$

Set 2: $V = \pm\sqrt{\frac{1+4\alpha\mu}{1+4\beta\mu}}, a_{-1} = \pm\frac{4(\alpha-\beta)\mu}{\sqrt{1+4\beta\mu}\sqrt{1+4\alpha\mu}}, a_0 = a_0,\ a_1 = 0, b_0 = 0, b_1 = 0, b_{-1} = 0,$

Set 3: $V = \pm\frac{\sqrt{1+\alpha\mu}}{\sqrt{1+\beta\mu}}, a_{-1} = \pm\frac{2(\alpha-\beta)\mu}{\sqrt{1+\alpha\mu}\sqrt{1+\beta\mu}}, a_0 = a_0,\ a_1 = 0, b_0 = \pm\frac{2(\alpha-\beta)\mu}{\sqrt{\sigma}\sqrt{1+\beta\mu}\sqrt{1+\alpha\mu}}, b_1 = 0, b_{-1} = 0$

Substituting Set 1–Set 3 into Equation (3.3) along with Equations (2.6)–(2.9), we get the following families of traveling wave solutions:

Hyperbolic function solutions: When $\mu < 0$, we get the following three families of hyperbolic function solutions.

Family 1: $u_{1,2} = a_0 \pm \frac{4(\alpha\mu\lambda^2-\beta\mu\lambda^2-\beta+\alpha)\sqrt{-\mu}\tanh(A+\sqrt{-\mu}\xi)}{\sqrt{1+4\beta\mu}\sqrt{1+4\alpha\mu}(1+\lambda\sqrt{-\mu}\tanh(A+\sqrt{-\mu}\xi))},$

$$u_{3,4} = a_0 \pm \frac{4(\alpha\mu\lambda^2-\beta\mu\lambda^2-\beta+\alpha)\sqrt{-\mu}\coth(A+\sqrt{-\mu}\xi)}{\sqrt{1+4\beta\mu}\sqrt{1+4\alpha\mu}(1+\lambda\sqrt{-\mu}\coth(A+\sqrt{-\mu}\xi))},$$

where $\xi = x \pm \sqrt{\frac{1+4\alpha\mu}{1+4\beta\mu}}t,$

Family 2: $u_{5,6} = a_0 \pm \frac{4(\alpha-\beta)(1+\lambda\sqrt{-\mu}\tanh(A+\sqrt{-\mu}\xi)}{\sqrt{1+4\beta\mu}\sqrt{1+4\alpha\mu}\sqrt{-\mu}\tanh(A+\sqrt{-\mu}\xi))},$

$$u_{7,8} = a_0 \pm \frac{4(\alpha-\beta)(1+\lambda\sqrt{-\mu}\coth(A+\sqrt{-\mu}\xi))}{\sqrt{1+4\beta\mu}\sqrt{1+4\alpha\mu}\sqrt{-\mu}\coth(A+\sqrt{-\mu}\xi)},$$

where $\xi = x \pm \sqrt{\frac{1+4\alpha\mu}{1+4\beta\mu}}t,$

Family 3: $u_{9,10} = a_0 \pm \frac{2(\alpha-\beta)\mu}{\sqrt{-\mu}\sqrt{1+\beta\mu}\sqrt{1+\alpha\mu}}(\coth(A+\sqrt{-\mu}\xi) + \lambda\sqrt{-\mu} + \operatorname{csc}h(A+\sqrt{-\mu}\xi)),$

$$u_{11,12} = a_0 \pm \frac{2(\alpha-\beta)\mu}{\sqrt{-\mu}\sqrt{1+\beta\mu}\sqrt{1+\alpha\mu}}(\tanh(A+\sqrt{-\mu}\xi) + \lambda\sqrt{-\mu} + I\,\operatorname{sec}h(A+\sqrt{-\mu}\xi)),$$

where $\xi = x \pm \frac{\sqrt{1+\alpha\mu}}{\sqrt{1+\beta\mu}}t,$

Trigonometric function solutions: when $\mu > 0$, we get the following three families of trigonometric function solutions.

Family 4: $u_{13,14} = a_0 \pm \frac{4(\alpha-\beta)(1+\mu\lambda^2)\sqrt{\mu}\tan(A-\sqrt{\mu}\xi)}{\sqrt{1+4\beta\mu}\sqrt{1+4\alpha\mu}(1+\lambda\sqrt{\mu}\tan(A-\sqrt{\mu}\xi))}$,

$$u_{15,16} = a_0 \pm \frac{4(\alpha-\beta)(1+\mu\lambda^2)\sqrt{\mu}\cot(A+\sqrt{\mu}\xi)}{\sqrt{1+4\beta\mu}\sqrt{1+4\alpha\mu}(1+\lambda\sqrt{\mu}\cot(A+\sqrt{\mu}\xi))},$$

where $\xi = x \pm \sqrt{\frac{1+4\alpha\mu}{1+4\beta\mu}}t$,

Family 5: $u_{17,18} = a_0 \pm \frac{4(\alpha-\beta)\sqrt{\mu}(1+\lambda\sqrt{\mu}\tan(A-\sqrt{\mu}\xi)}{\sqrt{1+4\beta\mu}\sqrt{1+4\alpha\mu}\tan(A-\sqrt{\mu}\xi))}$,

$$u_{19,20} = a_0 \pm \frac{4(\alpha-\beta)\sqrt{\mu}(1+\lambda\sqrt{\mu}\cot(A+\sqrt{\mu}\xi))}{\sqrt{1+4\beta\mu}\sqrt{1+4\alpha\mu}\cot(A+\sqrt{\mu}\xi)},$$

where $\xi = x \pm \sqrt{\frac{1+4\alpha\mu}{1+4\beta\mu}}t$,

Family 6: $u_{21,22} = a_0 \pm \frac{2(\alpha-\beta)\sqrt{\mu}}{\sqrt{1+\beta\mu}\sqrt{1+\alpha\mu}}(\cot(A-\sqrt{\mu}\xi)+\lambda\sqrt{\mu}+\csc(A-\sqrt{\mu}\xi))$,

$$u_{23,24} = a_0 \pm \frac{2(\alpha-\beta)\sqrt{\mu}}{\sqrt{1+\beta\mu}\sqrt{1+\alpha\mu}}(\tan(A+\sqrt{\mu}\xi)+\lambda\sqrt{\mu}+\sec(A+\sqrt{\mu}\xi)),$$

where $\xi = x \pm \frac{\sqrt{1+\alpha\mu}}{\sqrt{1+\beta\mu}}t$,

Remark: All the obtained solutions have been checked with maple by putting them back into the original equations and found correct.

4. Comparison

By using the modified simple equation (MSE) method, Akter and Akbar [36] achieved only eight exact solutions of the Benney–Luke equation (see Appendix A). On the contrary by using the enhanced (G'/G)-expansion method in this article we obtained 24 solutions. From our solutions obtained in this study, easily we can say that the enhanced (G'/G)-expansion method is very powerful, effective, and convenient. The performance of this method is reliable, simple, and gives many new solutions. Hence, the exact solutions of the Benney–Luke equation have reached in this article by using the computation software, Maple. Similarly for any nonlinear evolution equation it can be shown that the enhanced (G'/G)-expansion method is much powerful than other methods.

5. Discussion

In this section, we will discuss the physical interpretation and graphical representation of the solution of the Benney–Luke equation.

5.1. *Physical interpretation*

By applying the enhanced (G'/G)-expansion method to the Benney–Luke equation affords the traveling wave solutions are $u_{1,2}(x,t)$, $u_{3,4}(x,t)$, $u_{5,6}(x,t)$, $u_{7,8}(x,t)$, $u_{9,10}(x,t)$, $u_{11,12}(x,t)$

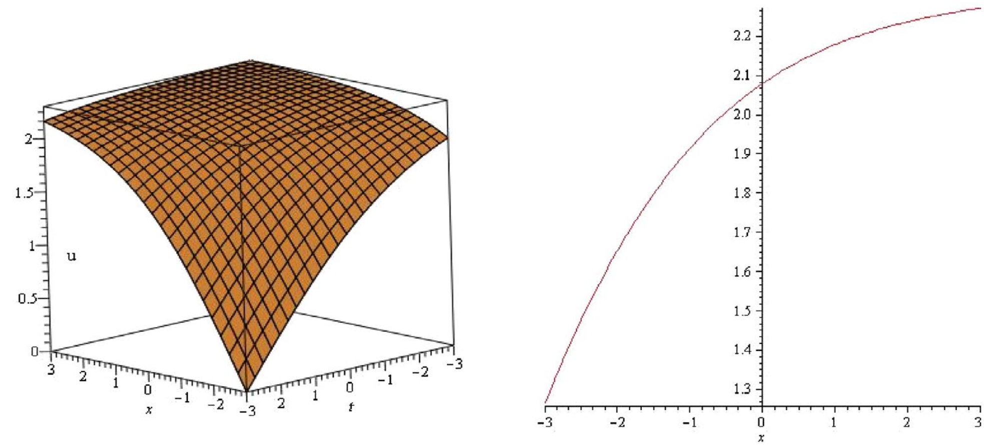

Figure 1. Anti 1-soliton wave profile of Benney–Luke equation for $a_0 = 1,\ \alpha = 2,\ \beta = 1,\ \mu = -.08,\ A = 1,\ \lambda = 1.5$ within the interval $-3 \le (x, t) \le 3$. (Only shows the shape of $u_1(x, t)$). The left figure shows the 3D plot and the right figure shows the 2D plot for $t = 0$.

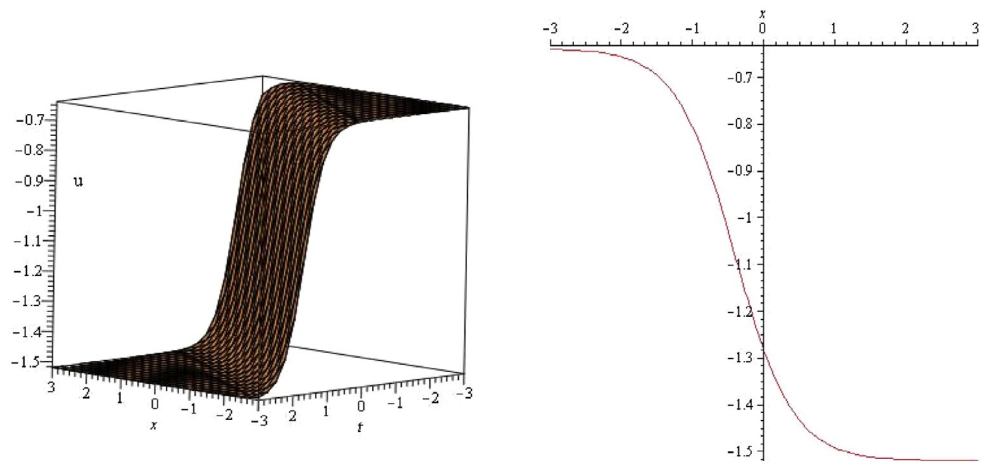

Figure 2. Kink type soliton profile of Benney Luke equation for $a_0 = 0,\ \alpha = 2,\ \beta = 1,\ \mu = -1.5,\ A = 2,\ \lambda = 0.5$ with $-3 \le (x, t) \le 3$. (Only shows the shape of $u_7(x, t)$), The left figure shows the 3D plot and the right figure shows the 2D plot for $t = 0$.

$u_{13,14}(x, t), u_{15,16}(x,t),\ u_{17,18}(x,t),\ u_{19,20}(x,t),\ u_{21,22}(x,t),\ u_{23,24}(x,t)$. The solution $u_1(x, t)$ is represented in Figure 1. It shows the shape of Anti 1-soliton solitary wave solution corresponding to the values of $a_0 = 1,\ \alpha = 2,\ \beta = 1,\ \mu = -.08,\ A = 1,\ \lambda = 1.5$ within the interval $-3 \le (x,t) \le 3$. Kink wave are traveling waves which rise or descent from one asymptotic state to another. The kink solution approaches a constant at infinity. Solution $u_7(x, t)$ corresponding to the fixed values of $a_0 = 0,\ \alpha = 2,\ \beta = 1,\ \mu = -1.5,\ A = 2,\ \lambda = 0.5$ with $-3 \le (x, t) \le 3$ represented the exact solitary wave solution of kink type. The exact kink type solitary wave solution is shown in Figure 2. Figure 3 represents the periodic traveling wave solution corresponding to $u_{21}(x, t)$ with fixed parameters as $a_0 = 1.4,\ \alpha = 2.5,\ \beta = 1,\ \mu = 5,\ A = 0.5,\ \lambda = 2.7$ and $-3 \le (x, t) \le 3$. Solution $u_{17}(x, t)$ is represented in Figure 4 which shows the shape of periodic solution with $a_0 = 0$, $\alpha = 0.02$, $\beta = 0.1$, $\mu = 2.7$, $A = 2.5$, $\lambda = 2.5$ within the interval $-3 \le (x, t) \le 3$. Finally, the solution $u_{23}(x, t)$ of the Benney–Luke equation represented in Figure 5 shows the shape of periodic wave solution for the unknown parameters $a_0 = 0$, $\alpha = 2.5$, $\beta = 1$, $\mu = 0.49$, $A = 1$, $\lambda = 1$within the interval $-3 \le (x, t) \le 3$.

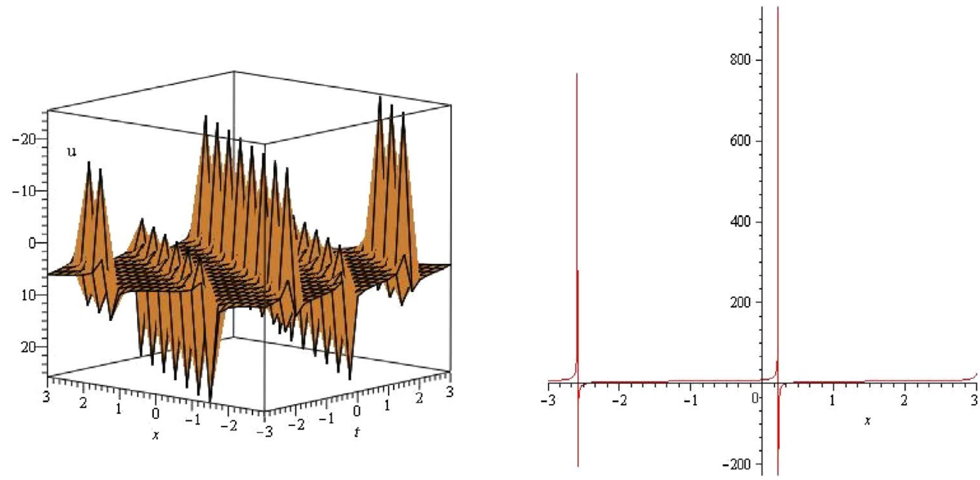

Figure 3. Periodic wave profile of Benney–Luke equation for $a_0 = 1.4$, $\alpha = 2.5$, $\beta = 1$, $\mu = 5$, $A = 0.5$, $\lambda = 2.7$ with $-3 \le (x, t) \le 3$. (Only shows the shape of $u_{21}(x, t)$), The left figure shows the 3D plot and the right figure shows the 2D plot for $t = 0$.

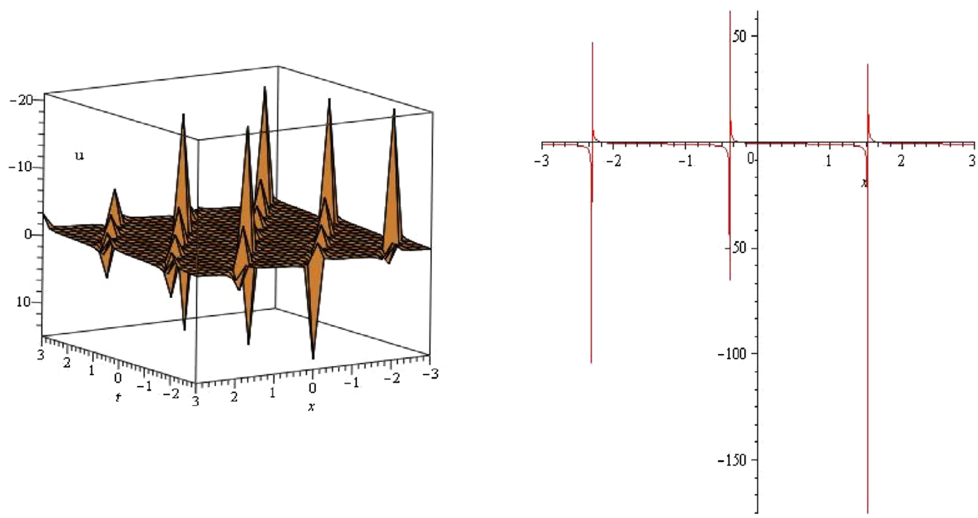

Figure 4. Periodic wave profile of Benney–Luke equation for $a_0 = 0$, $\alpha = 0.02$, $\beta = 0.1$, $\mu = 2.7$, $A = 2.5$, $\lambda = 2.5$ with $-3 \le (x, t) \le 3$. (Only shows the shape of $u_{17}(x, t)$), The left figure shows the 3D plot and the right figure shows the 2D plot for $t = 0$.

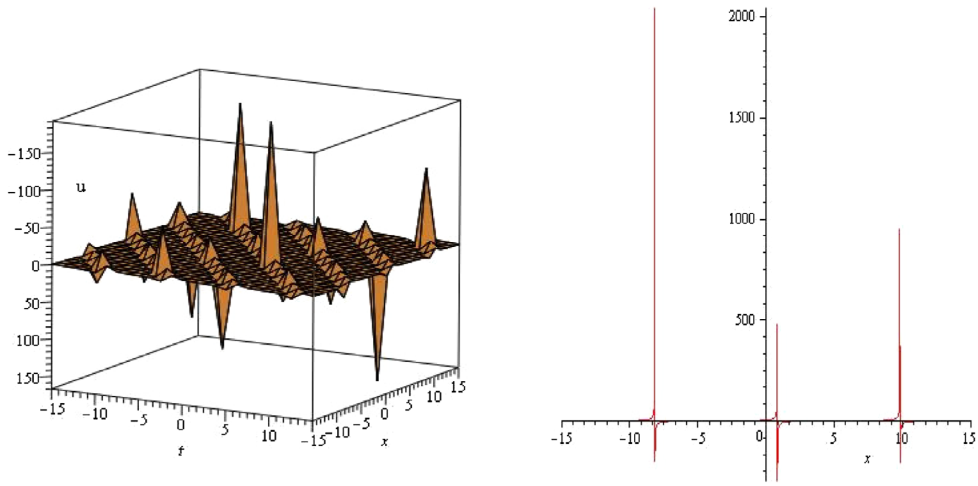

Figure 5. Periodic wave profile of Benney–Luke equation for $a_0 = 0,\ \alpha = 2.5,\ \beta = 1,\ \mu = 0.49,\ A = 1,\ \lambda = 1$ with $-15 \le (x, t) \le 15$. (Only shows the shape of $u_{17}(x, t)$), The left figure shows the 3D plot and the right figure shows the 2D plot for $t = 0$.

5.2. Graphical explanation

This sub-section presents the graphical representation of the Benney–Luke equation. By using mathematical software Maple 13, 2D and 3D plots of some achieved solutions have been shown in Figures 1–5 to envisage the essential instrument of the original equations.

6. Conclusion

In this paper, the enhanced (G'/G)-expansion method has been applied to the Benney–Luke equation. Then it has given us some new analytic solutions for Equation (1.1). All these analytic solutions founded in this study have been checked up by means of commercial software Maple 13. This method gives rise to hyperbolic and trigonometric function solutions. Under the terms of these information, it have been seen that enhanced (G'/G)-expansion method is a powerful tool for obtaining analytical solutions to Equation (1.1). We think that this technique can also be applied to other nonlinear evolution equations.

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ORCID

Kamruzzaman Khan http://orcid.org/0000-0002-4531-288X

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Appendix A

Akter and Akbar [36] obtained the exact solutions of the Benney–Luke equation by making use of the MSE method. They obtained solutions are given below:

$$u(x,t) = A_0 \mp \frac{2\sqrt{1-V^2}\sqrt{\alpha-\beta V^2}}{V}\left(1 \pm \tanh\left(\frac{1}{2}\sqrt{\frac{1-V^2}{\alpha-\beta V^2}}(x-Vt)\right)\right),$$

$$u(x,t) = A_0 \mp \frac{2\sqrt{1-V^2}\sqrt{\alpha-\beta V^2}}{V}\left(1 \pm \coth\left(\frac{1}{2}\sqrt{\frac{1-V^2}{\alpha-\beta V^2}}(x-Vt)\right)\right),$$

$$u(x,t) = A_0 \mp \frac{2\sqrt{1-V^2}\sqrt{\alpha-\beta V^2}}{V}\left(1 \mp i\tan\left(\frac{1}{2}\sqrt{\frac{1-V^2}{\alpha-\beta V^2}}\,i(x-Vt)\right)\right),$$

$$u(x,t) = A_0 \mp \frac{2\sqrt{1-V^2}\sqrt{\alpha-\beta V^2}}{V}\left(1 \pm i\cot\left(\frac{1}{2}\sqrt{\frac{1-V^2}{\alpha-\beta V^2}}\,i(x-Vt)\right)\right),$$